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**SCHEDULING RESERVOIR OIL PRODUCTION
BY LINEAR PROGRAMMING AND
THE NEYMAN-PEARSON LEMMA**

by
Michel Ruche

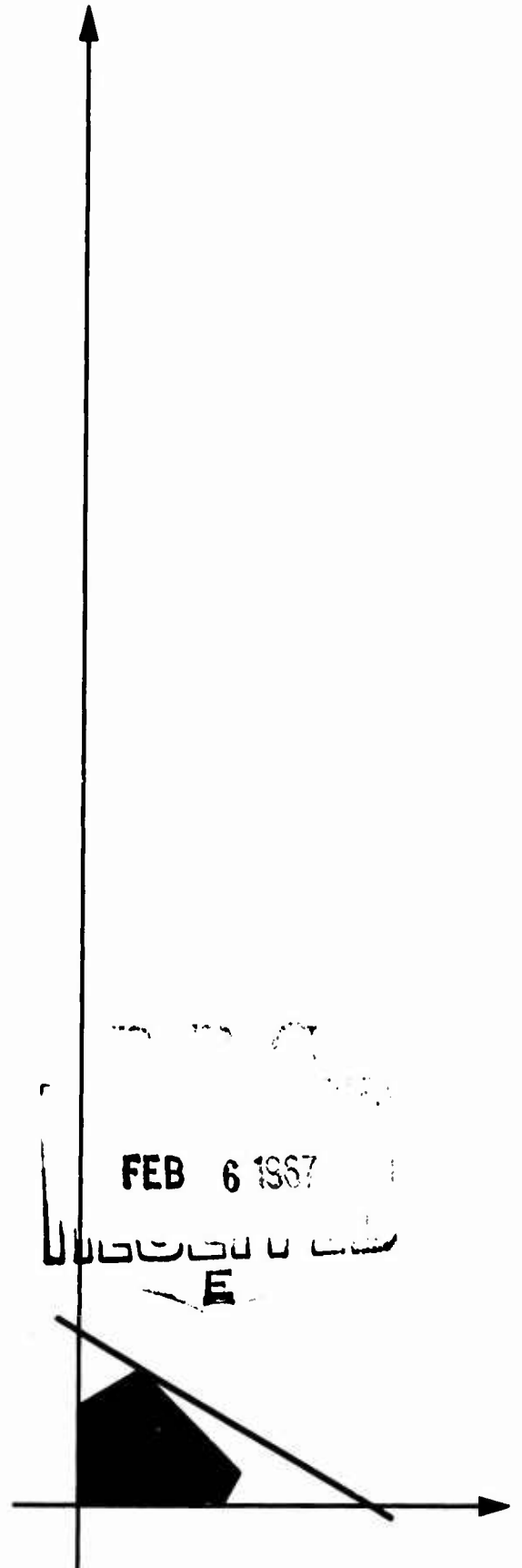
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SCHEDULING RESERVOIR OIL PRODUCTION BY LINEAR PROGRAMMING AND
THE NEYMAN-PEARSON LEMMA

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Scheduling Reservoir Oil Production by Linear Programming and
the Neyman-Pearson Lemma

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1. STATEMENT OF THE PROBLEM

In a set of n oil wells located in reservoirs without communications, we wish to schedule the flowrate of each individual well in such a way that we satisfy the following conditions:

- the pressure drop of each individual well is less than or equal to a given value,
- the production of any well is bounded by lower and upper technical limits,
- the total production of the fields has to satisfy a certain demand with some specific technical limiting capacities,
- the total discounted profit of the oil production is maximized.

Following the basic fluid and cash flow mechanisms of the system, two flowrate arrangements will be considered, namely:

- discrete flowrate,
- continuous flowrate.

They will be applied to systems including single or multiwells through a linear programming model and an attempt to use the Neyman-Pearson lemma.

1.1 FLUID FLOW

Besides limitations on the production of each part of a given oil field, a special consideration has to be made for the maximum pressure drop in the reservoir. Since the maximum is located at the bottom of the well bore, this region will have to satisfy at any time the pressure constraints.

The concept of a well function defined as the bottom hole pressure drop at time t_x per unit of constant flowrate is written as

$$F_j(t_x) \quad \text{where} \quad F_{j,i}(0) = 0$$

and due to the linearity of the diffusivity equation, we can use the superposition principle or convolution method as:

$$P_{O,j} - P_{x,j} = \sum_{i=1}^n (q_{i,j} - q_{i-1,j}) F_j(t_x - t_{i,j}) \quad 1.102$$

We want to satisfy for the well j ,

$$P_{O,j} - P_{x,j} \leq \delta_j \quad 1.103$$

therefore, in the case of a discrete flowrate,

$$\boxed{\sum_{i=1}^n (q_{i,j} - q_{i-1,j}) F_j(t_x - t_{i,j}) \leq \delta_j} \quad 1.104$$

and without loss of generality, we can assume

$$t_{i,j} - t_{i-1,j} = \Delta t = \frac{T}{n}$$

and

$$\sum_{i=1}^n \frac{q_{i,j} - q_{i-1,j}}{\Delta t} \Delta t F_j(t_x - t_{i,j}) \leq \delta_j \quad 1.105$$

or, at the limit as $\Delta t \rightarrow 0$ for a continuous flowrate,

$$\boxed{\int_{t=0}^{t=t_x} q_{i,j}'(t) F_j(t_x - t_{i,j}) dt \leq \delta_j} \quad 1.106$$

where

$$q_{i,j}'(t) = \lim_{\substack{n \rightarrow \infty \\ \text{or } \Delta t \rightarrow 0}} \left(\frac{q_{i,j} - q_{i-1,j}}{\Delta t} \right)$$

The three well functions which will be used in the calculation are represented in figs. 1 and 2.

Maximum and minimum limitations

Uniquely referring to the reservoir capacities with some pressure drop imposed, we can define two limitations on the flowrate, namely:

max Q_{\max} : constant flowrate during any time period which will produce the maximum permissible drop at the end of the period.

max Q_{\min} : constant flowrate during the entire time which will produce the maximum permissible pressure drop at the end of the entire period.

For a well j :

$$\max q_{\max} = \frac{\delta_j}{F_j\left(\frac{T}{n}\right)} \quad \text{and} \quad \max q_{\min} = \frac{\delta_j}{F_j(T)} \quad 1.107$$

For a multiwell system which includes m wells:

$$\max Q_{\max} = \sum_{j=1}^m \frac{\delta_j}{F_j\left(\frac{T}{n}\right)} \quad \text{and} \quad \max Q_{\min} = \sum_{j=1}^m \frac{\delta_j}{F_j(T)} \quad 1.108$$

The technical capacities and the demands must be such that:

$$\min(\max q_{\min}, \min \text{ technical capacities well } j) \leq \text{individual flowrate } q_{x,j} \quad 1.109$$

$$\min(\max q_{\max}, \max \text{ technical capacities well } j) \geq \text{individual flowrate } q_{x,j} \quad 1.110$$

$$\min(\max Q_{\min}, \min \text{ total technical capacities}) \leq \text{total flowrate } Q_x \quad 1.111$$

$$\min (\max Q_{\max}, \max \text{ total technical capacities} \geq \text{total flowrate } Q_x \quad 1.112$$

and

$$\text{demand} \leq \text{total flowrate} \quad 1.113$$

1.2 CASH FLOW

Between times t_i and t_{i+1} when the flowrate is $q_{i,j}$ the profit is such that:

$$\text{Profit} = \int_{t_i}^{t_{i+1}} e^{-\lambda T \tau} q_{i,j}(t) dt \quad 1.201$$

where the discount function is $e^{-\lambda T \tau}$ and where $\tau = \frac{t-x}{T}$

$$\text{Let} \quad \alpha = \lambda T \quad \text{and} \quad \gamma = e^{-\alpha}$$

$$\text{or} \quad \tau = 0 \Rightarrow e^{-\lambda T \tau} = 1$$

$$\text{and} \quad \tau = 1 \Rightarrow e^{-\lambda T \tau} = \gamma$$

Four cases will be considered here:

$$\gamma_0 = 1.00 \Rightarrow \alpha_0 = 0.000$$

$$\gamma_1 = 0.90 \Rightarrow \alpha_1 = 0.105$$

$$\gamma_2 = 0.75 \Rightarrow \alpha_2 = 0.288$$

$$\gamma_3 = 0.50 \Rightarrow \alpha_3 = 0.693$$

The corresponding discount functions are shown on fig. 3.

Total discounted profit

a) Discrete flowrate: assuming a constant flowrate $q_{i,j}(t)$ between t_i and t_{i+1} :

$$\text{Profit} = \sum_{i=1}^n \int_{t_i}^{t_{i+1}} e^{-\frac{\alpha t}{T}} q_{i,j}(t) dt \quad 1.202$$

$$\text{Total discounted profit} = \sum_{\substack{i=1 \\ n \in T}}^n q_{i,j}(t) \frac{T}{\alpha} (e^{-\alpha \tau_i} - e^{-\alpha \tau_{i+1}}) \quad 1.203$$

Let $T = n\Delta t$ and $\tau_i^* = \frac{\tau_i + \tau_{i+1}}{2}$

then

$$\text{Total discounted profit} = \frac{T}{n} \sum_{i=1}^n e^{-\alpha \tau_i^*} q_{i,j}(t) \quad 1.204$$

b) Continuous flowrate:

$$\text{Total discounted profit} = \int_0^T e^{-\frac{\alpha t}{T}} q_{i,j}(t) dt \quad 1.205$$

The profit function has the dimension of dollars and it represents the total profit over the entire lifetime of the project.

2. INDIVIDUAL WELL PROBLEM

We consider one well in a reservoir and it is required that the oil production of the given well be scheduled during a time T . In a first linear program model, the total time T is divided into a series of n time periods Δt and $q_{i,j}(t)$ is computed. Then, the Neyman-Pearson lemma will be applied in the case of a continuous flowrate for the determination of the exact function $q_j(t)$.

It is obvious that, in both cases, the cumulative production, namely,

$$\sum_{\substack{i=1 \\ n \in T}}^n q_{i,j}(t) \Delta t_i$$

or

$$\int_0^T q_j(t) dt$$

cannot be greater than the reserves of the reservoir. However, this

last constraint will not be considered here, and we assume that for any well j we satisfy:

$$\sum_{i=1}^n q_{i,j}(t) \Delta t_i \leq R_j \quad \text{or} \quad \int_0^T q_j(t) dt \leq R_j \quad 2.001$$

where R_j are the reserves for the well j .

2.1 DISCRETE FLOWRATE

From the preceding statements, in the case of a discrete flowrate the function $q_{i,j}(t)$ is the solution of the following system:

$$1) \text{ maximize } \frac{T}{n} \sum_{i=1}^n e^{-\alpha \tau_i^*} q_{i,j}(t) \quad 2.101$$

over all functions $q_{i,j}(t)$ subject to

2) the pressure constraints:

$$\sum_{i=1}^n q_{i,j}(t) \left\{ F_j \left[(N - i + 1) \Delta t \right] - F_j \left[(N - i) \Delta t \right] \right\} \leq \delta_j \quad 2.102$$

$N = 1, \dots, n$

or

$$\sum_{i=1}^n q_{i,j}(t) \left\{ \bar{\omega}_j \left[(N - i + 1) \Delta t \right] - \bar{\omega}_j \left[(N - i) \Delta t \right] \right\} \leq \Delta_j \quad 2.103$$

where $\Delta_j = \frac{2\pi h k}{\eta} \delta_j$

3) bounded by some technical limiting capacities as:

$$q_{i,j}^{\min}(t) \leq q_{i,j}(t) \leq q_{i,j}^{\max}(t) \quad 2.104$$

4) where the entire lifetime of the project is defined as:

$$0 \leq t_i \leq T \quad 2.105$$

This problem is easily solved by the use of computers and well-improved linear programming techniques. The solutions obtained, however, show a dependence under the selected value of n . The true maximum could be obtained by greatly increasing n . But the size of our problem is fast becoming large and lengthy. It is therefore interesting to try to determine the actual true maximum by another method dealing with continuous flowrates.

2.2 CONTINUOUS FLOWRATE

In the case of a continuous flowrate, the function $q_j(t)$ is the solution of the following system:

$$1) \text{ maximize } \int_0^T e^{-\frac{\alpha t}{T}} q_j(t) dt \quad 2.201$$

over all functions $q_j(t)$ subject to

2) the pressure constraint:

$$\int_0^t q_j'(t) \bar{w}_j(t_x - t) dt \leq \Delta_j \quad 2.202$$

3) bounded by some technical limiting capacities, as

$$q_{j,\min} \leq q_j(t) \leq q_{j,\max} \quad 2.203$$

4) where the entire lifetime of the project is defined as

$$0 \leq t \leq T \quad 2.204$$

The above system will be somewhat modified through a change of variables, in order to take advantage of the Neyman-Pearson lemma as presented in appendix 5.1 and by fig. 14.

$$\text{Let} \quad \tau = \frac{t}{T}$$

$$\text{then} \quad 0 \leq \tau \leq 1 \quad 2.205$$

$$\text{and} \quad q_j(\tau) = \frac{q_j(\tau T)}{q_{j,\max}} \quad 2.206$$

$$\text{or} \quad 0 \leq q_j(\tau) \leq 1 \quad 2.207$$

Now consider the equation 2.201 which is equivalent to

$$\text{minimize} \int_0^T -e^{-\alpha \frac{t}{T}} q_j(t) dt$$

Or, if we let

$$a(\tau) = -e^{-\alpha \tau} \quad 2.208$$

the objective function of the system is

$$\text{minimize} \int_0^1 a(\tau) q_j(\tau) d\tau \quad 2.209$$

Consider the function $I(t)$ defined as

$$I(t) = \int_0^t q_j'(t) \bar{w}_j(T - t) dt$$

and let us transform this function in the following way:

$$I_j(\tau) = \int_0^\tau q_j'(T\tau) \bar{w}_j[T(1 - \tau)] T d\tau$$

$$\frac{dq_j(\tau)}{d\tau} = \frac{1}{q_{j,\max}} q_j'(T\tau) t_x \Rightarrow q_j'(T\tau) = \frac{dq_j(\tau)}{d\tau} \cdot \frac{q_{j,\max}}{T}$$

and, finally:

$$I_1(\tau) = q_{j,\max} \int_0^\tau \frac{dq_j(\tau)}{d\tau} \bar{\omega}_j[T(1 - \tau)] d\tau \quad 2.210$$

Integrating by parts, equation 2.202 can be stated as

$$\left\{ q_j(\tau) \bar{\omega}_j[T(1 - \tau)] \right\} \Big|_0^\tau - \int_0^\tau q_j(\tau) \frac{d\bar{\omega}_j[T(1 - \tau)]}{d\tau} d\tau \leq \frac{\Delta_j}{q_{j,\max}} \quad 2.211$$

Let

$$b(\tau) = - \frac{d\bar{\omega}_j[T(1 - \tau)]}{d\tau} \quad 2.212$$

and

$$c(\tau) = \frac{\Delta_j}{q_{j,\max}} - \left\{ q_j(\tau) \bar{\omega}_j[T(1 - \tau)] \right\} \Big|_0^\tau \quad 2.213$$

or

$$c(\tau) = \frac{\Delta_j}{q_{j,\max}} + q_j(0) \bar{\omega}_j(T) - q_j(\tau) \bar{\omega}_j[T(1 - \tau)] \quad 2.214$$

Our initial system is now the following:

$$\text{minimize } \int_0^1 a(\tau) q_j(\tau) d\tau \quad 2.215$$

over all functions $q_j(t)$ subject to the constraints:

$$a) \quad 0 \leq q_j(\tau) \leq 1, \quad 0 \leq \tau \leq 1 \quad 2.216$$

$$b) \quad \int_0^\tau b(\tau) q_j(\tau) d\tau \leq c(\tau) \quad 2.217$$

where $a(\tau)$, $b(\tau)$, and $c(\tau)$ are given functions such as

$$a(\tau) = -e^{-\alpha\tau} \quad 2.218$$

$$b(\tau) = - \frac{d\bar{\omega}_j \left[T(1 - \tau) \right]}{d\tau} \quad 2.219$$

$$c(\tau) = \frac{\Delta_j}{q_{j\max}} + q_j(0)\bar{\omega}_j(T) - q_j(\tau)\bar{\omega}_j \left[T(1 - \tau) \right] \quad 2.220$$

This form of the system allows free use of the Neyman-Pearson lemma, from which we obtain the optimum solution.

2.3 PROPOSED SOLUTIONS

A numerical solution is now presented for the case of an individual well surrounded by a radial circular reservoir. The derivation of the well function of such a system is shown in appendix 5.2. Three different wells will be considered. In this first part only one single well, say well j , is included in the system that we want to schedule.

All the units in this study are expressed in terms of the centimeter-gram-second system, the results being shown in practical units.

Characteristics of the wells

- 1) thickness of the formation: $h = 10^3$ cm
- 2) well radius: $a = 10$ cm
- 3) outside reservoir radius: $R = 2 \times 10^5$ cm or $R \geq 1$ mile
- 4) porosity: $\omega = 10\%$
- 5) permeability:
 - well (1): $k = 10^{-12}$ CGS or 0.1 millidarcy
 - well (2): $k = 10^{-11}$ CGS or 1.0 millidarcy
 - well (3): $k = 10^{-10}$ CGS or 10.0 millidarcy

6) compressibility of the fluids: $\beta = 10^{-10}$ CGS (cm^2 per dyne)

7) viscosity of the fluids: $\mu = 10^{-2}$ CGS (Poise or dyne-second per cm^2)

The maximum pressure drop allowed here is of 10^8 dynes or 1,450 psi.

The numerical solution presented covers a time interval of 1 year. As can be seen, the flowrate profiles are independent of the economic conditions imposed in the system.

2.31 DISCRETE FLOWRATE LINEAR PROGRAMMING MODEL

We want to solve equations 2.101, 2.102, and 2.103 for a well j , where

$$T = 1 \text{ year} \quad \text{and} \quad n = 12$$

Well function

In this case the function $\bar{w}_3(\theta)$ (see appendix 5.2, equation 5.211) can be used:

$$\bar{w}_3(\theta) = \frac{1}{2}(\text{Log } \theta + 0.80907)$$

Or let

$$F_j(t) = \bar{a} \log(t) + \bar{b} \quad 2.3101$$

where:

	\bar{a}	\bar{b}
well (1)	1.83239×10^6	$- 1.18852 \times 10^6$
well (2)	1.83239×10^5	0.6438×10^5
well (3)	1.83239×10^4	2.4762×10^4

Maximum and minimum flowrates

From equation 1.107 and when $\delta_j = 10^8$ dynes:

	$F_j(T)$	$F_j(T/6)$	$F_j(T/12)$	Maximum (CGS)	Max q_{\max}		
					n = 1	n = 6	n = 12
well (1)	12.541×10^6	11.115×10^6	10.564×10^6	7.974	7.974	8.997	9.466
well (2)	14.373×10^5	12.947×10^5	12.396×10^5	69.575	69.575	77.238	80.671
well (3)	16.206×10^4	14.780×10^4	14.229×10^4	617.055	617.055	676.589	702.790

Assuming that the technical capacities are such that

$$\text{minimum technical capacities} = 0.0$$

and

$$\text{maximum technical capacities} > \max q_{j,\max}$$

equations 1.109 and 1.110 are reduced to

$$0 \leq \text{flowrate } q_j \leq \max q_{j,\max} \quad 2.3102$$

Solution

The solution of this simple linear programming problem is easily obtained by using the Share program number HOSCM3 on the 7094 IBM computer.

In fig. 4 a case is presented where upper and lower limits exist on the individual flowrates for well (2), assuming four different discount functions, namely, $\alpha_0, \alpha_1, \alpha_2, \alpha_3$, as defined in section 1.2.

The optimum flowrate expressed as the ratio $q_{\text{average}}/q_{\max}$ for different values of q_{\max}/q_{\max} is shown on fig. 5 for the three wells. The profit is also represented, and the influence of the number n on the final results can be seen.

2.32 CONTINUOUS FLOWRATE NEYMAN-PEARSON LEMMA

From equations 5.201 and 2.219,

$$b(\tau) = 2 \sum_{n=1}^4 \alpha_n \beta_n^2 \left(\frac{a^2}{K}\right) e^{-\frac{\beta_n^2 a^2}{K} T(1-\tau)} \quad 2.3201$$

from equation 2.220,

$$\begin{aligned} c(\tau) = & \frac{\Delta_j}{q_{j\max}} + q_j(0) \left[\text{Log } R' - 2 \sum \alpha_n e^{-\frac{\beta_n^2 a^2}{K} T} \right] \\ & - q_j(\tau) \left[\text{Log } R' - 2 \sum \alpha_n e^{-\frac{\beta_n^2 a^2}{K} T(1-\tau)} \right] \end{aligned} \quad 2.3202$$

and from equation 5.1104,

$$A(\tau) = 2 \left[\sum \alpha_n e^{-\frac{\beta_n^2 a^2}{K} T(1-\tau)} - \sum \alpha_n e^{-\frac{\beta_n^2 a^2}{K} T} \right] \quad 2.3203$$

These three functions, $b(\tau)$, $c(\tau)$, and $A(\tau)$ are well defined for a given well and present some interesting properties. For instance, $b(\tau)$ being a monotonically non-negative increasing function with a finite value when $\tau = 0$, and $a(\tau)$ having also a finite value when $\tau = 0$, the supremum value of α_0' , as defined in appendix 5.1, is such that

$$\alpha_0' = -\frac{e^{-a}}{b(1)} \quad 2.3204$$

Therefore, τ_1 , the intersection of $a(\tau)$ and $\alpha_0' b(\tau)$, is equal to one.

Thus a given well will be flowing at its maximum flowrate from $\tau = 0$ to $\tau = \tau_2$, τ_2 being defined as the intersection of $A(\tau)$ and $c(\tau)$. Then it is only necessary to determine τ_2 and to verify that $A(\tau) \geq c(\tau)$ when $0 \leq \tau \leq \tau_2$.

If we notice also that $A(0) = 0$ and $c(0) = \frac{\Delta_j}{q_{j,\max}} > 0$ then

$$q_j(0) = 1 \quad 2.3205$$

This means that the well has to be set at its maximum value at zero time. This maximum value will remain up to the time τ_2 , and the intersection of $A(\tau)$ and $c(\tau)$ can be determined by setting $q_j(\tau) = 1$ in equation 2.3202. However, a careful observation of the problem shows that we have to satisfy equation 2.217, where one of the limits is a function of τ . Therefore, in some cases, it could be hazardous to apply the Neyman-Pearson lemma systematically. A more complete method of optimal control, as the maximum principle of Pontriagine, would be preferable (ref. 5,6). This limitation restricts, to a certain extent, a simple application of the Neyman-Pearson lemma for this particular problem.

3 MULTIWELL SYSTEM

The multiwell system is considered here as an extension of the individual case when m wells are producing from a given set of reservoirs. When more than one well acts on the same system, however, interference effects are frequently observed. These effects will not be taken into account here; only the simplified case where all the wells are independent will be developed, assuming that each well has the same effect as if it were alone in the system.

Therefore, the cumulative production will consist of the linear summation of all the wells and the total cash flow will mediate the corresponding production with its economics.

3.1 DISCRETE FLOWRATE

Initially, it was thought that a generalization of the fundamental lemma of Neyman and Pearson might be used for the treatment of the multiwell system (ref. 4). However, due to the above-mentioned limitations regarding the individual well, this method had to be used with caution. It would be more advisable to use the optimum control principles presented in references 5 or 6.

Thus, this study will be concerned only with some aspects of the discrete case where the problem to be solved is the following:

-- Find the optimum flowrate of a multiwell system including m wells, with the following constraints:

- pressure drop maximum at every well
- upper and lower bounds on the wells
- a certain demand to be satisfied

Then the formulation of the problem can be set as:

$$1) \text{ maximize } \sum_{j=1}^m \frac{T}{n} \sum_{i=1}^n e^{-\alpha \tau_i^*} q_{i,j}(t) \quad 3.1$$

over all functions $q_{i,j}(t)$ subject to

$$2) \quad \sum_{j=1}^m \sum_{i=1}^n q_{i,j}(t) \left\{ \bar{\omega}_j \left[(N - i + 1) \Delta t \right] - \bar{\omega}_j \left[(N - i) \Delta t \right] \right\} \leq \Delta_j$$

$$N = 1, \dots, n \quad 3.2$$

3) bounded by some technical limiting capacities as

$$q_{i,j,\min}(t) \leq q_{i,j}(t) \leq q_{i,j,\max}(t) \quad 3.3$$

$$4) \quad \sum_{j=1}^m q_{1,j}(t) \leq Q(t) \quad 3.4$$

5) where the entire lifetime of the project is defined as

$$0 \leq t \leq T \quad 3.5$$

3.2 PROPOSED SOLUTION: LINEAR PROGRAMMING MODEL

This linear programming model has been solved for different cases.

The basic matrix is presented in fig. 6, where:

$$n = 6, \quad m = 3, \quad q_{1,j}^{\min} = 0$$

$$q_{1,j}^{\max} = \infty, \quad Q(t) = \text{constant}, \quad \Delta_j = 10^8 \text{ dynes}$$

Due to the number of constraints, the size of the matrix increases as n is increased. So, when

$n = 1$ we get a matrix of 8 rows \times 16 columns

$n = 6$ we get a matrix of 28 rows \times 46 columns

$n = 12$ we get a matrix of 52 rows \times 82 columns

and when n is large, the problem is greatly simplified by using the decomposition principle for linear programming.

Also, the number of rows and columns in the basic matrix determines whether the primal or the dual type of problem should be used, the fewer the number of rows in the matrix, the easier to solve the problem.

An obvious simplification can be obtained by fixing a priori $n = 1$. As can be seen, however, a true maximum is not attained by this method. But due to the small increase of the total cash flow (a few per cent) when using $n = 6$ instead of $n = 1$, this simplification can be worthwhile. It

can also be noted that by changing the value of n , it is possible to modify the well recovery to a considerable extent. This is illustrated by fig. 8, case B_3 , and fig. 13, case B_2 .

Two categories of results are shown from figs. 7 to 13.

1) Constant output-equality cases where

$$\sum_{j=1}^m q_{i,j}(t) = \text{constant (in arbitrary units)} \quad 3.6$$

a) no limitations on individual flowrate

fig. 7 $n = 6$

fig. 12 $n = 12$

b) limitations on individual flowrate

fig. 11 $n = 6$

2) Total bounded output-inequality cases where

$$\sum_{j=1}^m q_{i,j}(t) \leq \text{constant (in arbitrary units)} \quad 3.7$$

figs. 8, 9, 10 $n = 6$

fig. 13 $n = 12$

The results are shown with the following characteristics:

-- on the left side of the page, the economics based on the α -coefficient values

-- in the center of the page, the production profile

-- on the right side of the page, the total and individual output for the three wells

Four basic mechanisms have been observed:

A. For the constant output-equality cases, an improvement of the economics leads to an increase of the cash flow associated with a decrease of the recovery from each well.

B. For the total bounded output-inequality cases we fix a constant demand and follow the change of the optimum flowrates for different values of the constant. The interesting parameters are the decrease of the demand and the change of the economics.

Then:

-- for large values of the demand, all the wells have to flow to their maximum;

-- when decreasing the demand, the best wells are kept at their maximum value while the other wells are reduced conversely proportional to their productivity;

-- for a given value of the demand corresponding to a maximum constant demand for the whole period, the situation is completely modified;

-- as soon as the demand is below the previous value, the bad wells are fixed to their maxima and the final fit is made on the best well in reference to its productivity.

This procedure can be followed on figs. 8, 9, 10, and 13. On figs. 7 and 12, the decrease of the maximum value of a constant demand is shown to be a function of the different economics imposed on the wells.

C. Effects of upper and lower bounds are shown on fig. 11, which can be compared with fig. 7.

D. Production profile. During the entire period the well production has to satisfy a certain profile. The first steps of this profile are especially interesting.

The results shown can be considered as typical behavior for a three-well system where

well (1) is a low productivity well

well (2) is a medium productivity well

well (3) is a high productivity well

The different values of the economics must be sufficient to cover most of the actual cases.

Using an IBM 709⁴ computer and Share program number HOSCM3, the computing time was 2 seconds for each case when $n = 6$.

4 CONCLUSION

-- A good estimate of the optimum cash flow can be obtained with $r = 10$.

-- The optimum recovery of a well is very sensitive to the economics imposed on the system, as, for example, the lower and upper bounds restrictions.

-- The bad wells must produce at a flowrate proportionately higher than the best wells.

-- The production profile of the well is not obvious in most cases, but an accurate approach to the optimum profile can be obtained at moderate cost by using a linear programming model.

5 APPENDIX5.1 THE NEYMAN-PEARSON LEMMA5.11 Statement of the problem

We wish to minimize a linear functional of the form:

$$\int_0^1 f(\tau) a(\tau) d\tau \quad 5.1101$$

over all functions $f(\tau)$ subject to the following constraints:

$$a) \ 0 \leq f(\tau) \leq 1, \ 0 \leq \tau \leq 1 \quad 5.1102$$

$$b) \ \int_0^1 f(\tau) b(\tau) d\tau \leq c \quad 5.1103$$

where $a(\tau) \leq 0$ and $b(\tau) \geq 0$ are given functions, and $c \geq 0$ is a known constant.

5.12 Solution

The solution is determined as follows. Let set functions

$$E^- = E^-(\alpha'), \quad E = E(\alpha'), \quad E^+ = E^+(\alpha')$$

be defined for $-\infty < \alpha < +\infty$ as:

$$E^-(\alpha') = [\tau; a(\tau) < \alpha' b(\tau)]$$

$$E(\alpha') = [\tau; a(\tau) = \alpha' b(\tau)]$$

$$E^+(\alpha') = [\tau; a(\tau) > \alpha' b(\tau)]$$

Determine α_0' by the condition that α_0' be the supremum over all non-positive α' satisfying the inequality

$$\int_{E^-} b(\tau) d\tau \leq c \quad 5.1104$$

and let

$$E^-(\alpha_0') = E_0^-$$

$$E(\alpha_0') = E_0$$

$$E^+(\alpha_0') = E_0^+$$

Then the set of minimizing functions f^* is given by:

$$a) f^*(\tau) = 1 \text{ on } E_0^-$$

$$b) f^*(\tau) = 0 \text{ on } E_0^+$$

$$c) f^*(\tau) = \text{arbitrary on } E_0, \text{ satisfying only the condition}$$

$$0 \leq f^*(\tau) \leq 1, \quad 0 \leq \tau \leq 1$$

and

$$\int_0^1 f^*(\tau) b(\tau) d\tau = c \quad \text{if } \alpha_0' < 0$$

or the conditions

$$0 \leq f^*(\tau) \leq 1, \quad 0 \leq \tau \leq 1 \quad \text{if } \alpha_0' = 0$$

5.13 Outline of the procedure

$$a) \text{ choose } \alpha' = \alpha_i' < 0$$

$$b) \text{ find } E^-(\alpha_i') = [\tau; a(\tau) < \alpha_i' b(\tau)]$$

$$c) \text{ compute } A = \int_{E^-} b(\tau) d\tau$$

$$d) \text{ if } A(\tau) > c, \text{ decrease } \alpha_i' \text{ and go to (a)}$$

$$\text{if } A(\tau) < c, \text{ increase } \alpha_i' \text{ and go to (a)}$$

Optimal α_i' is α_0' such that for $\alpha_i' + \Delta\alpha_i$, $A(\tau) > c$. See fig. 14.

5.2 WELL FUNCTION OF THE SINGLE WELL

The well function, defined as the pressure drop in a given well corresponding to a constant unit flowrate at time t , can be computed in many ways (see, for instance, ref. 2). Here we are interested in the analytical formulation of the well function corresponding to a radial system, and we show that such a function can be written as:

$$\bar{w}_0(\theta) = \text{Log } R' - 2 \sum_{n=1}^4 \alpha_n e^{-\beta_n^2 \epsilon} \quad 5.201$$

where each term is defined below.

Radial flow of a homogeneous fluid into a single well

The well completely penetrates the reservoir and the fluid flows uniformly in all directions radiating to the axis of the well bore. The system has axial symmetry and we can describe the fluid pressure variations by the Laplace equation in cylindrical coordinates, namely:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} = \frac{1}{K} \frac{\partial p}{\partial t} \quad 5.202$$

with dimensionless variables as follows:

$$\text{distances: } r' = \frac{r}{a}$$

$$\text{time: } \epsilon = \frac{Kt}{a^2}$$

$$\text{pressure: } \bar{w} = (P_G - P) \frac{2\pi hk}{\eta Q}$$

Equation 5.202 is then reduced to:

$$\frac{\partial^2 \bar{w}}{\partial r'^2} + \frac{1}{r'} \frac{\partial \bar{w}}{\partial r'} = \frac{\partial \bar{w}}{\partial \theta} \quad 5.203$$

Initial conditions

-- constant pressure in the whole reservoir at the start, or:

$$\theta = 0, \quad \bar{w} = 0 \quad 5.204$$

-- boundary conditions:

constant flowrate from the well, or:

$$\left(\frac{\partial \bar{w}}{\partial r'} \right)_{r'=1} = -1 \quad \text{for any } \theta \quad 5.205$$

no flowrate from the outside border and constant pressure at the reservoir limit, or:

$$\bar{w} = 0 \quad \text{for any } \theta \quad 5.206$$

Analytical solution

The solution of the above system with the given initial and boundary conditions is such that:

$$\bar{w}_1(\theta) = \text{Log } R' - 2 \sum_{n=1}^{\infty} \frac{J_0^2(\beta_n R') e^{-\beta_n^2 \theta}}{\beta_n^2 [J_1^2(\beta_n) - J_0^2(\beta_n R')]} \quad 5.207$$

where the β_n 's are roots of:

$$Y_0(\beta_n R') J_1(\beta_n) - J_0(\beta_n R') Y_1(\beta_n) = 0 \quad 5.208$$

If we notice that the last term of 5.207 decreases rapidly as the time increases, after a certain time:

$$\bar{\omega}_1(\theta) = \text{Log } R' \quad 5.209$$

Equation 5.207 represents the unsteady state and equation 5.208 represents the steady state. $\bar{\omega}_1(\theta)$ is shown on fig. 15.

Approximations $\bar{\omega}_2(\theta)$, $\bar{\omega}_3(\theta)$ and $\bar{\omega}_4(\theta)$

On fig. 15 we can differentiate three zones, namely:

- zone 1: the initial part of the curve,
- zone 2: the central part presenting a straight semilog variation,
- zone 3: the end portion with the asymptotic value $\text{Log } R'$.

These three zones are bounded by times θ_1 and θ_2 which will be considered further.

If we now vary R' , we see on fig. 16 that all the curves representing $\bar{\omega}_1 \theta$ are tangent to the curve $\bar{\omega}_2(\theta)$ for which the external radius is infinite. Physically this property is very easy to justify. Then the calculus shows that when the radius is infinite:

$$\bar{\omega}_2(\theta) = \frac{4}{\pi^2} \int_0^\infty \frac{1 - e^{-u^2\theta}}{u^3 [J_1^2(u) + Y_1^2(u)]} du \quad 5.210$$

and for values of θ larger than θ_1 this last expression converges to

$$\bar{\omega}_3(\theta) = \frac{1}{2}(\text{Log } \theta + 0.80907) \quad 5.211$$

The semilogarithmic variation for the case of a reservoir of finite radius can be used up to a certain value θ_2 which will be considered further.

If we now assume that the well radius is null and the outside radius infinite, the solution is such that:

$$\bar{w}_4(\theta) = \frac{1}{2} E_1\left(-\frac{1}{4\theta}\right) \quad 5.212$$

This last function can be of some use when we want to compute the well function in an interval such that $\theta \leq \theta_1$. Besides its obvious simplicity compared with $\bar{w}_1(\theta)$ or $\bar{w}_2(\theta)$, an interesting use of the $\bar{w}_0(\theta)$ function can be seen through its derivative when referring to the Neyman-Pearson method, for instance.

Interval for the use of $\bar{w}_3(\theta)$: θ_1 and θ_2

θ_1 : the convergence of $\bar{w}_1(\theta)$ to $\bar{w}_3(\theta)$ is illustrated on fig. 17, where we can see that the approximation $\bar{w}_3(\theta)$ for $\bar{w}_1(\theta)$ is excellent as far as $\theta \geq 10^2$ and this limit is practically independent of R' .

θ_2 : the value of θ_2 is defined as the limiting value of θ up to which we get an appropriate convergence between $\bar{w}_1(\theta)$ and $\bar{w}_3(\theta)$. If we fix a ratio of convergence such that:

$$\frac{\bar{w}_1(\theta) - \bar{w}_3(\theta)}{\bar{w}_1(\theta)} = 6/1000 \quad 5.213$$

we get fig. 18, from where:

$$\text{Log } \theta_2 = a \text{ Log } R' + b \quad 5.214$$

which shows that θ_2 is dependent on R' under a simple formulation.

Calculation of α_n and β_n

α_n : from equation 5.207,

$$\alpha_n = \frac{J_0^2(\beta_n R')}{\beta_n^2 [J_1^2(\beta_n) - J_0^2(\beta_n R')]} = \frac{1}{\beta_n^2 \left[\frac{J_1^2(\beta_n)}{J_0^2(\beta_n R')} - 1 \right]} \quad 5.215$$

but from 5.208,

$$\frac{J_1^2(\beta_n)}{J_0^2(\beta_n R')} = \frac{Y_1^2(\beta_n)}{Y_0^2(\beta_n R')} \quad 5.216$$

and for small values of β_n

$$\text{Log } Y_1(\beta_n) = - \text{Log } \beta_n + \beta_a$$

or

$$\left(Y_1 \right)_x = \left(Y_1 \right)_a \frac{a}{x} \quad 5.217$$

when $R' > 100$ $\begin{cases} Y_1 \text{ is large, say } > 600 \\ Y_0(\beta R') \text{ is small, say } < 1 \end{cases}$

$$\boxed{\alpha_n = 2.46738 \left[Y_0(\beta_n R') \right]^2} \quad 5.218$$

and for large values of n in order to satisfy 5.204, from where

$$\text{Log } R' = 2 \sum_{n=1}^{\infty} \alpha_n \quad 5.219$$

we get

$$\boxed{\alpha_n = \frac{1}{2n} \left(1 - \frac{1}{R'}\right)^n} \quad n \geq 15 \quad 5.220$$

which is practically independent of R' as is shown on fig. 19, when, for instance, R' is larger than 100, this being a reasonable assumption for a practical case.

β_n : the β_n 's shown on fig. 20 are the roots of 5.203 and easy to compute from:

$$\frac{J_1(\beta_n)}{Y_1(\beta_n)} = \frac{J_0(\beta_n R')}{Y_0(\beta_n R')} \quad 5.221$$

When $R' > 100$, β_n is small and $J_1(0) = 0$; then $J_1(\beta_n) \simeq 0$, and

$$\boxed{J_0(\beta_n R') = 0} \quad 5.222$$

As soon as n is larger than 15, the positive roots of 5.222 are given approximately by:

$$\boxed{\beta_n = \frac{\pi}{R'} \left(n - \frac{1}{4}\right)} \quad n \geq 15 \quad 5.223$$

The following table summarizes the values of α_n and $\beta_n R'$ ($R' \geq 100$) obtained from the above considerations.

Table 1

n	α_n	$\beta_n R'$	n	α_n	$\beta_n R'$	n	α_n	$\beta_n R'$
1	0.6415	2.405	11	0.0465	33.76	21	0.0241	65.18
2	0.2834	5.520	12	0.0425	36.92	22	0.0230	68.33
3	0.1812	8.654	13	0.0392	40.06	23	0.0220	71.47
4	0.1331	11.79	14	0.0364	43.20	24	0.0211	74.62
5	0.1051	14.93	15	0.0339	46.34	25	0.0202	77.76
6	0.0869	18.07	16	0.0317	49.48	26	0.0194	80.90
7	0.0740	21.21	17	0.0299	52.62	27	0.0187	84.04
8	0.0645	24.35	18	0.0281	55.77	28	0.0180	87.18
9	0.0571	27.50	19	0.0267	58.90	29	0.0174	90.32
10	0.0512	30.63	20	0.0253	62.05	30	0.0168	93.46

The asymptotic well function $\bar{w}_0(\theta)$

In some particular cases, and for a specific interval of time, the use of $\bar{w}_2(\theta)$, $\bar{w}_3(\theta)$, or $\bar{w}_4(\theta)$ can simplify the problem. This is especially observed when using the function $\bar{w}_3(\theta)$ between θ_1 and θ_2 . This last interval is sufficiently large in actual so that a semilogarithm variation is justified almost a priori for a given well. Nevertheless, it seems interesting to refer to a well function covering the complete time scale. Such a function can be estimated from the computed coefficients α_n and β_n . Only finite values of n will be used in order to simplify the formulation without altering its effective accuracy. Due to its property this function can be named the asymptotic well function $\bar{w}_0(\theta)$.

A computation of $\bar{\omega}_0(\theta)$ using the α_n and β_n coefficients from table 1 shows that the results reproduce those presented in reference 7 and that for $R' > 100$, the convergence of the number of terms is the following:

$$\theta = 1 \quad n = 81$$

$$\theta = 10 \quad n = 29$$

$$\theta = 10^2 \quad n = 10$$

$$\theta = 10^3 \quad n = 4$$

with a rapid convergence to one for larger values of θ depending on the actual value of R' .

It is then suggested that only the first four values of n be introduced for obtaining a good estimation of the well function of a single well located at the center of a circular reservoir.

NOMENCLATURE

- a : well radius
 $a(\tau)$: function (see 2.208)
 \bar{b} : coefficient
 $b(\tau)$: function (see 2.212)
 $c(\tau)$: function (see 2.213)
 $E_1(-t)$: exponential integral
 $f_j(\tau)$: modified pressure drop at well j for modified time τ
 $F_j(t)$: pressure drop at well j for a constant flowrate during time t
 h : reservoir thickness
 i : integer = 1, 2, ..., n , $n + 1$
 $I_1(\tau)$: function (see 2.210)
 j : integer = 1, 2, ..., $m - 1$, m
 k : reservoir permeability
 K : hydraulic diffusivity
 m : number of wells in the multiwell system
 n : number of equal intervals in period T
 N : integer = 1, ..., n
 $P_{0,j}$: initial pressure at well j
 $P_{i,j}$: pressure at well j , time t_i
 $P_j(t)$: pressure at well j , time t
 $P_{x,j}$: pressure at well j , time x
 $q_{i,j}$: discrete flowrate at well j , time t_i
 $q_j(t)$: continuous flowrate at well j , time t
 $q_j'(t)$: derivative of flowrate at well j , time t
 $q_{j\max}$: maximum flowrate allowed on well j
 $q_{j\min}$: minimum flowrate allowed on well j
 $q_{x,j}$: flowrate at well j , time t_x

$q_{x,j}'$: derivative of flowrate at well j , time t_x

$Q(t)$: total flowrate of the reservoir at time t

Q_{T_m} : total production of the well m

r' : reduced well radius

R : reservoir radius

R_j : reserves for well j

t : time $0 \leq t \leq T$

T : total period under consideration

t_i : time at which the flowrate q_i starts $0 \leq t_i \leq T$

t_x : time $0 \leq t_x \leq T$

y : integer = 1, 2, ..., n

Y_0, Y_{11}, J_0, J_1 : Bessel functions

α : function

α_n : roots (see 5.220)

β : oil compressibility

β_n : roots (see 5.223)

γ : coefficient

δ_j : actual pressure drop allowed at well j

Δ_j : modified pressure drop allowed at well j

Δ_t : interval of time

θ : reduced time

λ : coefficient

μ : oil viscosity

τ : modified time

τ_i^* : average modified time

ω : reservoir porosity

$\bar{\omega}(\theta)$: well function

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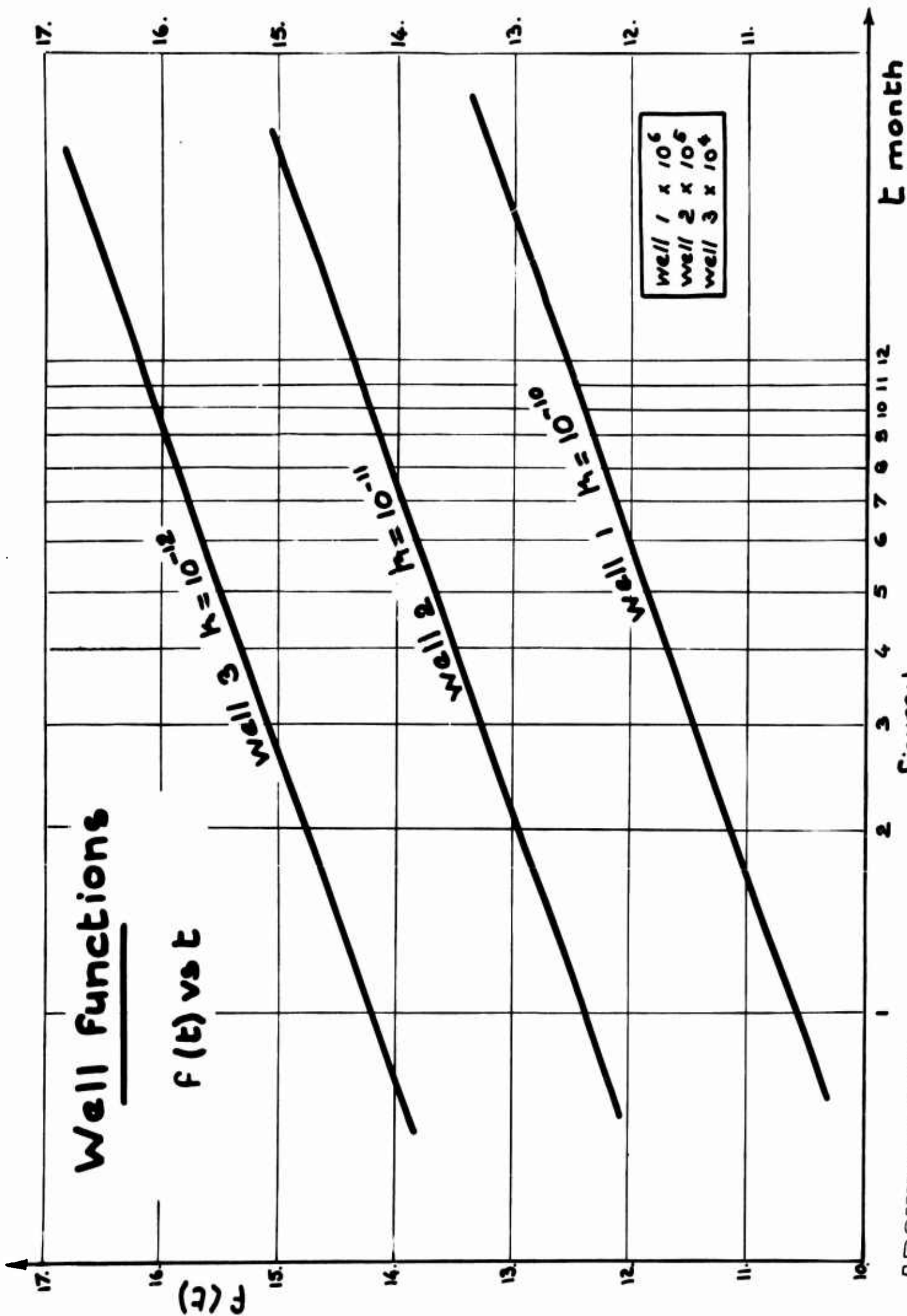


Figure:1

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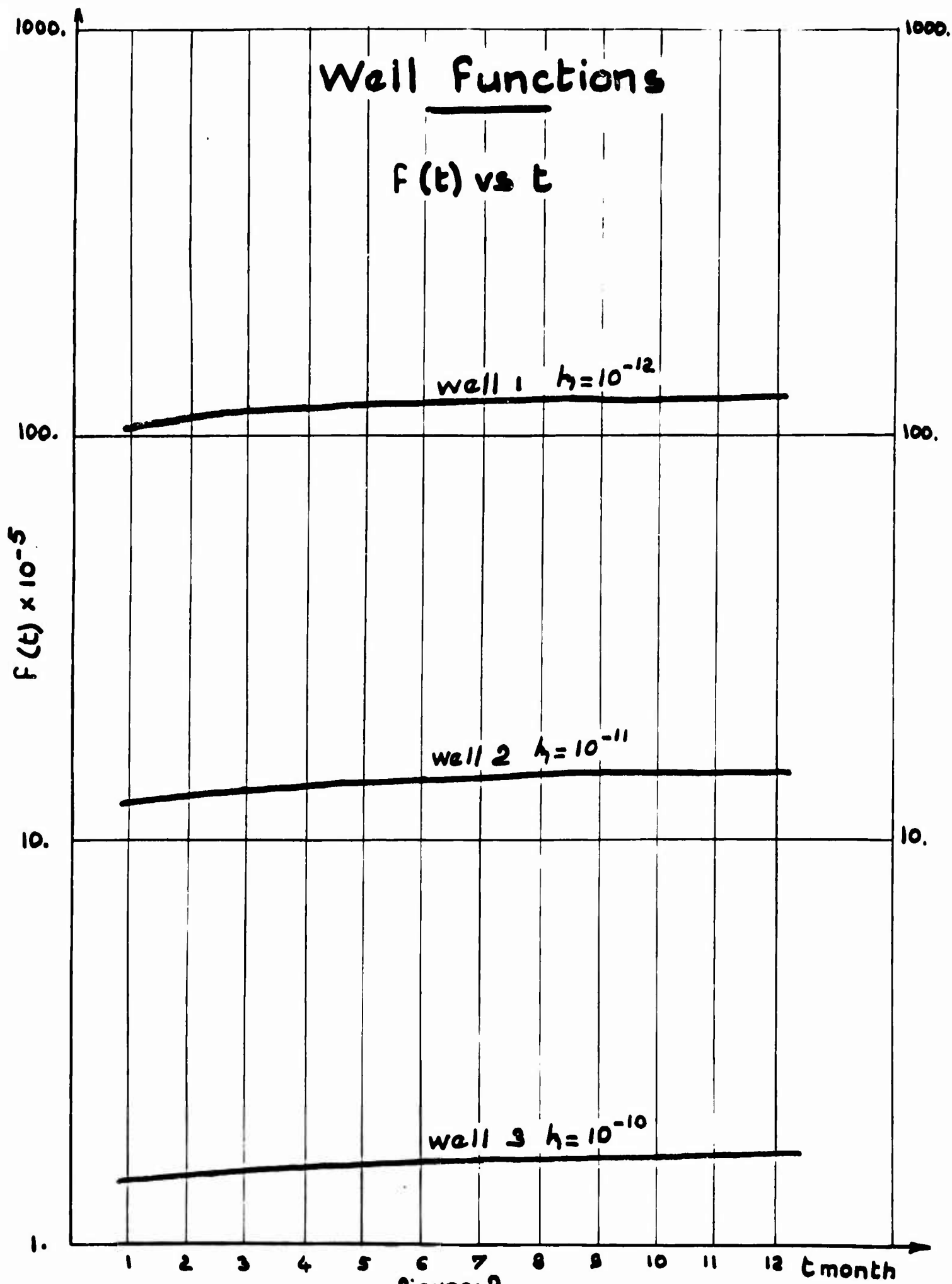


Figure: 2

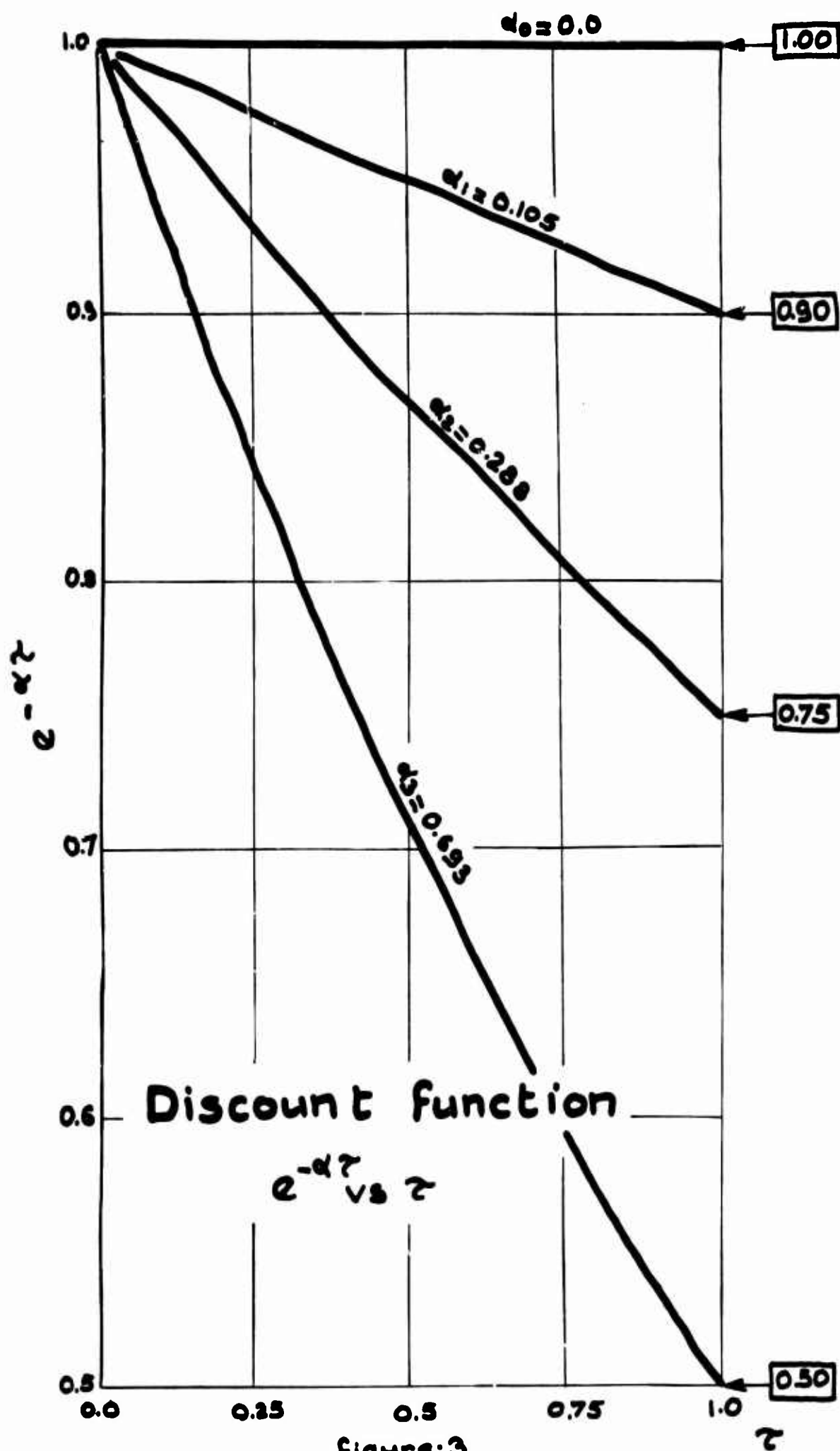


Figure:3

Single well matrix $n=12$ (limitations on individual flowrates)

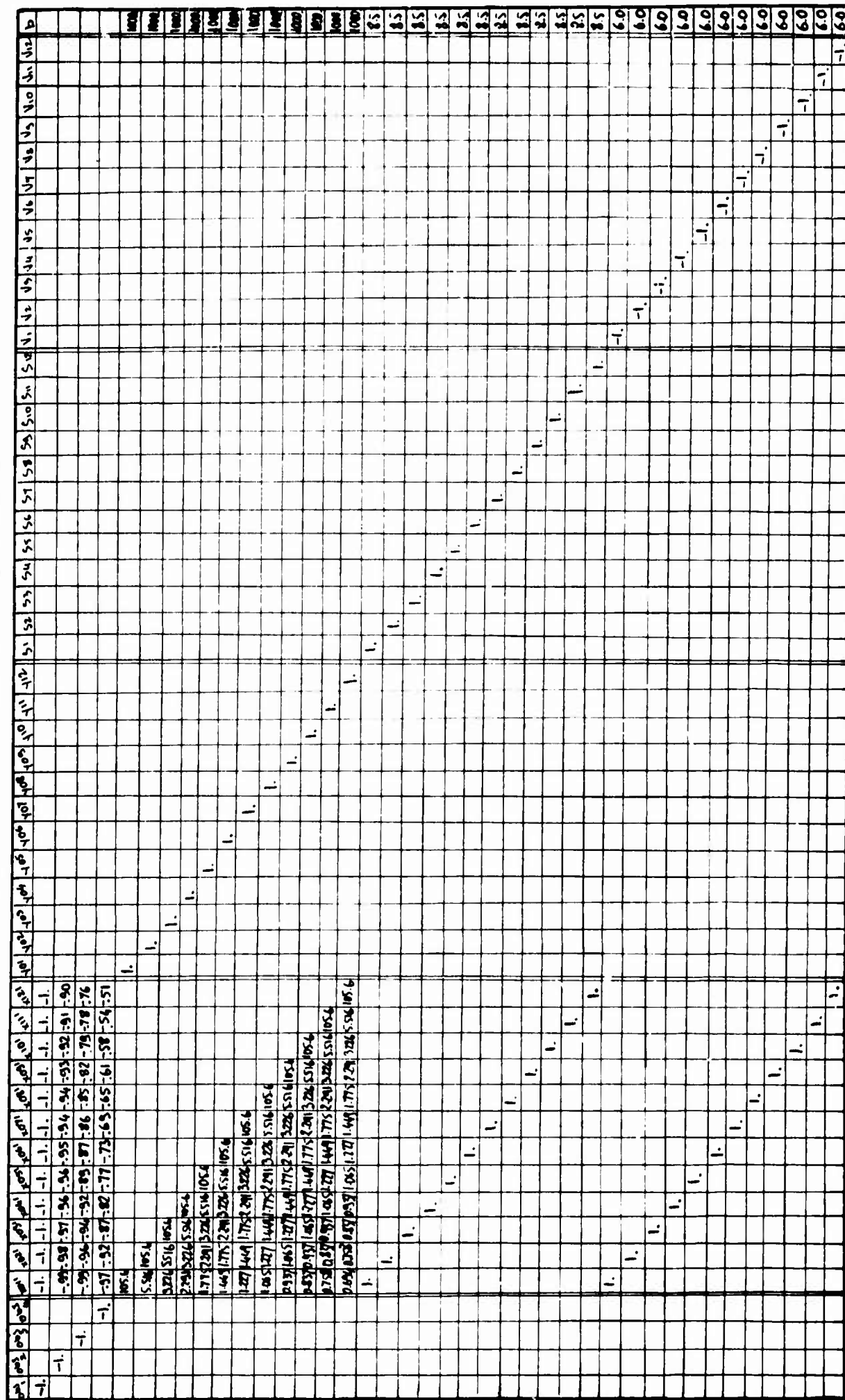


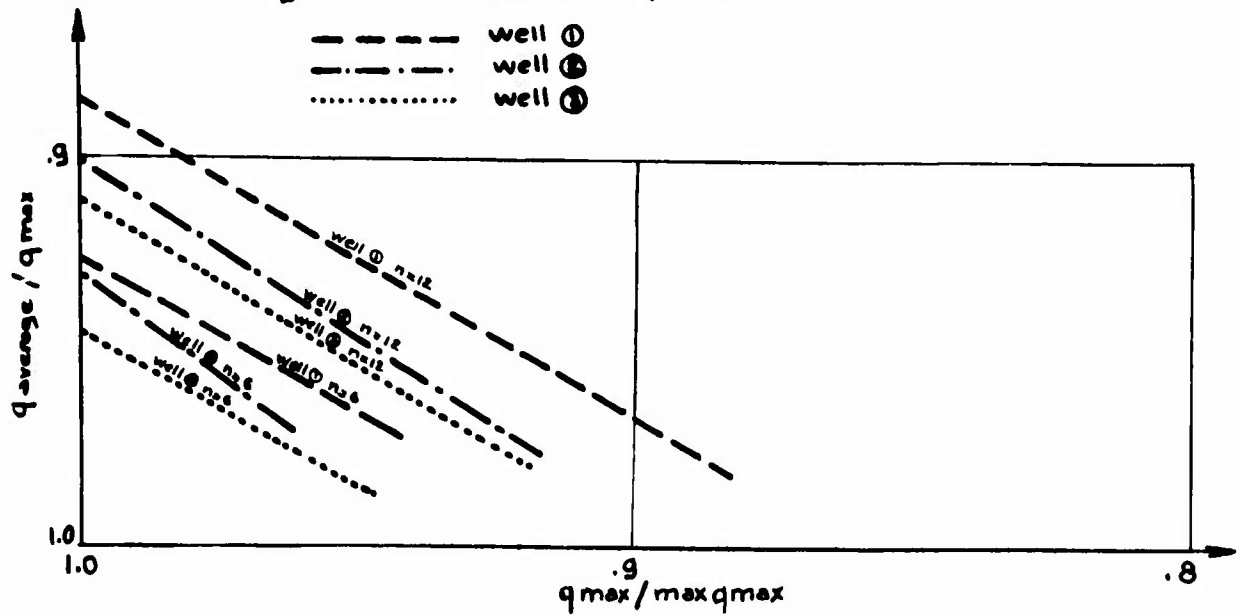
Figure : 4

Single well

Flowrat

($q_{max} \leq \max q_{max}$)

independent of α
 maxi at the beginning of the period
 mini at the end of the period



Profit: ($q_{max} \gg \max q_{max}$)

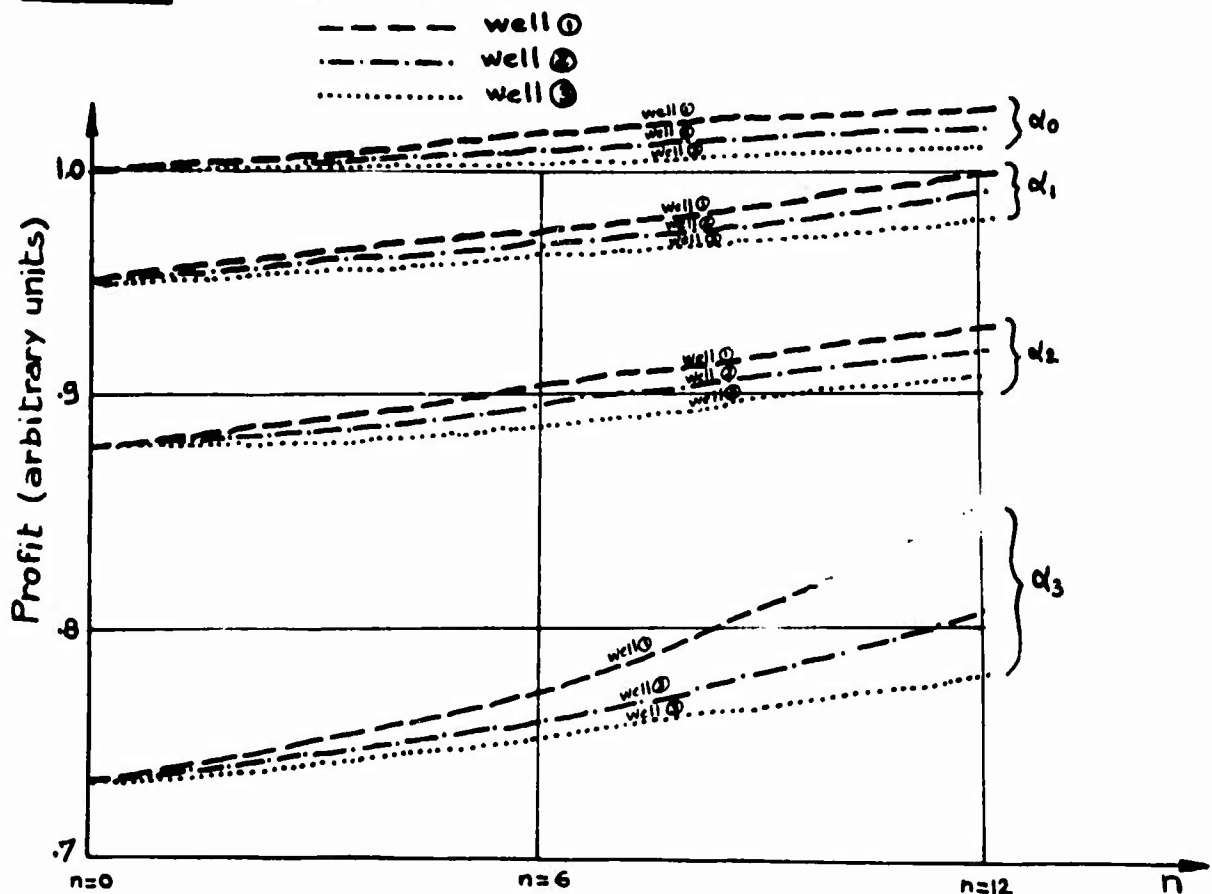


Figure: 5

Multiwell matrix - Equality cases n=6 (no limitations on individual flowrates)

X01	X02	X03	X04	X05	X06	X07	X08	X09	X10	X11	X12	X13	X14	X15	X16	X17	X18	X19	X20	X21	X22	X23	X24	X25	X26	X27	X28	X29	X30	X31	X32	X33	X34	X35	X36	X37	X38	X39	X40	X41	X42	X43	X44	X45	X46	X47	X48	X49	X50	X51	X52	X53	X54	X55	X56	X57	X58	X59	X60	X61	X62	X63	X64	X65	X66	X67	X68	X69	X70	X71	X72	X73	X74	X75	X76	X77	X78	X79	X80	X81	X82	X83	X84	X85	X86	X87	X88	X89	X90	X91	X92	X93	X94	X95	X96	X97	X98	X99	X100	X101	X102	X103	X104	X105	X106	X107	X108	X109	X110	X111	X112	X113	X114	X115	X116	X117	X118	X119	X120	X121	X122	X123	X124	X125	X126	X127	X128	X129	X130	X131	X132	X133	X134	X135	X136	X137	X138	X139	X140	X141	X142	X143	X144	X145	X146	X147	X148	X149	X150	X151	X152	X153	X154	X155	X156	X157	X158	X159	X160	X161	X162	X163	X164	X165	X166	X167	X168	X169	X170	X171	X172	X173	X174	X175	X176	X177	X178	X179	X180	X181	X182	X183	X184	X185	X186	X187	X188	X189	X190	X191	X192	X193	X194	X195	X196	X197	X198	X199	X200	X201	X202	X203	X204	X205	X206	X207	X208	X209	X210	X211	X212	X213	X214	X215	X216	X217	X218	X219	X220	X221	X222	X223	X224	X225	X226	X227	X228	X229	X230	X231	X232	X233	X234	X235	X236	X237	X238	X239	X240	X241	X242	X243	X244	X245	X246	X247	X248	X249	X250	X251	X252	X253	X254	X255	X256	X257	X258	X259	X260	X261	X262	X263	X264	X265	X266	X267	X268	X269	X270	X271	X272	X273	X274	X275	X276	X277	X278	X279	X280	X281	X282	X283	X284	X285	X286	X287	X288	X289	X290	X291	X292	X293	X294	X295	X296	X297	X298	X299	X300	X301	X302	X303	X304	X305	X306	X307	X308	X309	X310	X311	X312	X313	X314	X315	X316	X317	X318	X319	X320	X321	X322	X323	X324	X325	X326	X327	X328	X329	X330	X331	X332	X333	X334	X335	X336	X337	X338	X339	X340	X341	X342	X343	X344	X345	X346	X347	X348	X349	X350	X351	X352	X353	X354	X355	X356	X357	X358	X359	X360	X361	X362	X363	X364	X365	X366	X367	X368	X369	X370	X371	X372	X373	X374	X375	X376	X377	X378	X379	X380	X381	X382	X383	X384	X385	X386	X387	X388	X389	X390	X391	X392	X393	X394	X395	X396	X397	X398	X399	X400	X401	X402	X403	X404	X405	X406	X407	X408	X409	X410	X411	X412	X413	X414	X415	X416	X417	X418	X419	X420	X421	X422	X423	X424	X425	X426	X427	X428	X429	X430	X431	X432	X433	X434	X435	X436	X437	X438	X439	X440	X441	X442	X443	X444	X445	X446	X447	X448	X449	X450	X451	X452	X453	X454	X455	X456	X457	X458	X459	X460	X461	X462	X463	X464	X465	X466	X467	X468	X469	X470	X471	X472	X473	X474	X475	X476	X477	X478	X479	X480	X481	X482	X483	X484	X485	X486	X487	X488	X489	X490	X491	X492	X493	X494	X495	X496	X497	X498	X499	X500	X501	X502	X503	X504	X505	X506	X507	X508	X509	X510	X511	X512	X513	X514	X515	X516	X517	X518	X519	X520	X521	X522	X523	X524	X525	X526	X527	X528	X529	X530	X531	X532	X533	X534	X535	X536	X537	X538	X539	X540	X541	X542	X543	X544	X545	X546	X547	X548	X549	X550	X551	X552	X553	X554	X555	X556	X557	X558	X559	X560	X561	X562	X563	X564	X565	X566	X567	X568	X569	X570	X571	X572	X573	X574	X575	X576	X577	X578	X579	X580	X581	X582	X583	X584	X585	X586	X587	X588	X589	X590	X591	X592	X593	X594	X595	X596	X597	X598	X599	X600	X601	X602	X603	X604	X605	X606	X607	X608	X609	X610	X611	X612	X613	X614	X615	X616	X617	X618	X619	X620	X621	X622	X623	X624	X625	X626	X627	X628	X629	X630	X631	X632	X633	X634	X635	X636	X637	X638	X639	X640	X641	X642	X643	X644	X645	X646	X647	X648	X649	X650	X651	X652	X653	X654	X655	X656	X657	X658	X659	X660	X661	X662	X663	X664	X665	X666	X667	X668	X669	X670	X671	X672	X673	X674	X675	X676	X677	X678	X679	X680	X681	X682	X683	X684	X685	X686	X687	X688	X689	X690	X691	X692	X693	X694	X695	X696	X697	X698	X699	X700	X701	X702	X703	X704	X705	X706	X707	X708	X709	X710	X711	X712	X713	X714	X715	X716	X717	X718	X719	X720	X721	X722	X723	X724	X725	X726	X727	X728	X729	X730	X731	X732	X733	X734	X735	X736	X737	X738	X739	X740	X741	X742	X743	X744	X745	X746	X747	X748	X749	X750	X751	X752	X753	X754	X755	X756	X757	X758	X759	X760	X761	X762	X763	X764	X765	X766	X767	X768	X769	X770	X771	X772	X773	X774	X775	X776	X777	X778	X779	X780	X781	X782	X783	X784	X785	X786	X787	X788	X789	X790	X791	X792	X793	X794	X795	X796	X797	X798	X799	X800	X801	X802	X803	X804	X805	X806	X807	X808	X809	X810	X811	X812	X813	X814	X815	X816	X817	X818	X819	X820	X821	X822	X823	X824	X825	X826	X827	X828	X829	X830	X831	X832	X833	X834	X835	X836	X837	X838	X839	X840	X841	X842	X843	X844	X845	X846	X847	X848	X849	X850	X851	X852	X853	X854	X855	X856	X857	X858	X859	X860	X861	X862	X863	X864	X865	X866	X867	X868	X869	X870	X871	X872	X873	X874	X875	X876	X877	X878	X879	X880	X881	X882	X883	X884	X885	X886	X887	X888	X889	X890	X891	X892	X893	X894	X895	X896	X897	X898	X899	X900	X901	X902	X903	X904	X905	X906	X907	X908	X909	X910	X911	X912	X913	X914	X915	X916	X917	X918	X919	X920	X921	X922	X923	X924	X925	X926	X927	X928	X929	X930	X931	X932	X933	X934	X935	X936	X937	X938	X939	X940	X941	X942	X943	X944	X945	X946	X947	X948	X949	X950	X951	X952	X953	X954	X955	X956	X957	X958	X959	X960	X961	X962	X963	X964	X965	X966	X967	X968	X969	X970	X971	X972	X973	X974	X975	X976	X977	X978	X979	X980	X981	X982	X983	X984	X985	X986	X987	X988	X989	X990	X991	X992	X993	X994	X995	X996	X997	X998	X999	X1000	X1001	X1002	X1003	X1004	X1005	X1006	X1007	X1008	X1009	X1010	X1011	X1012	X1013	X1014	X1015	X1016	X1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Total constant output - Equality cases $n=6$

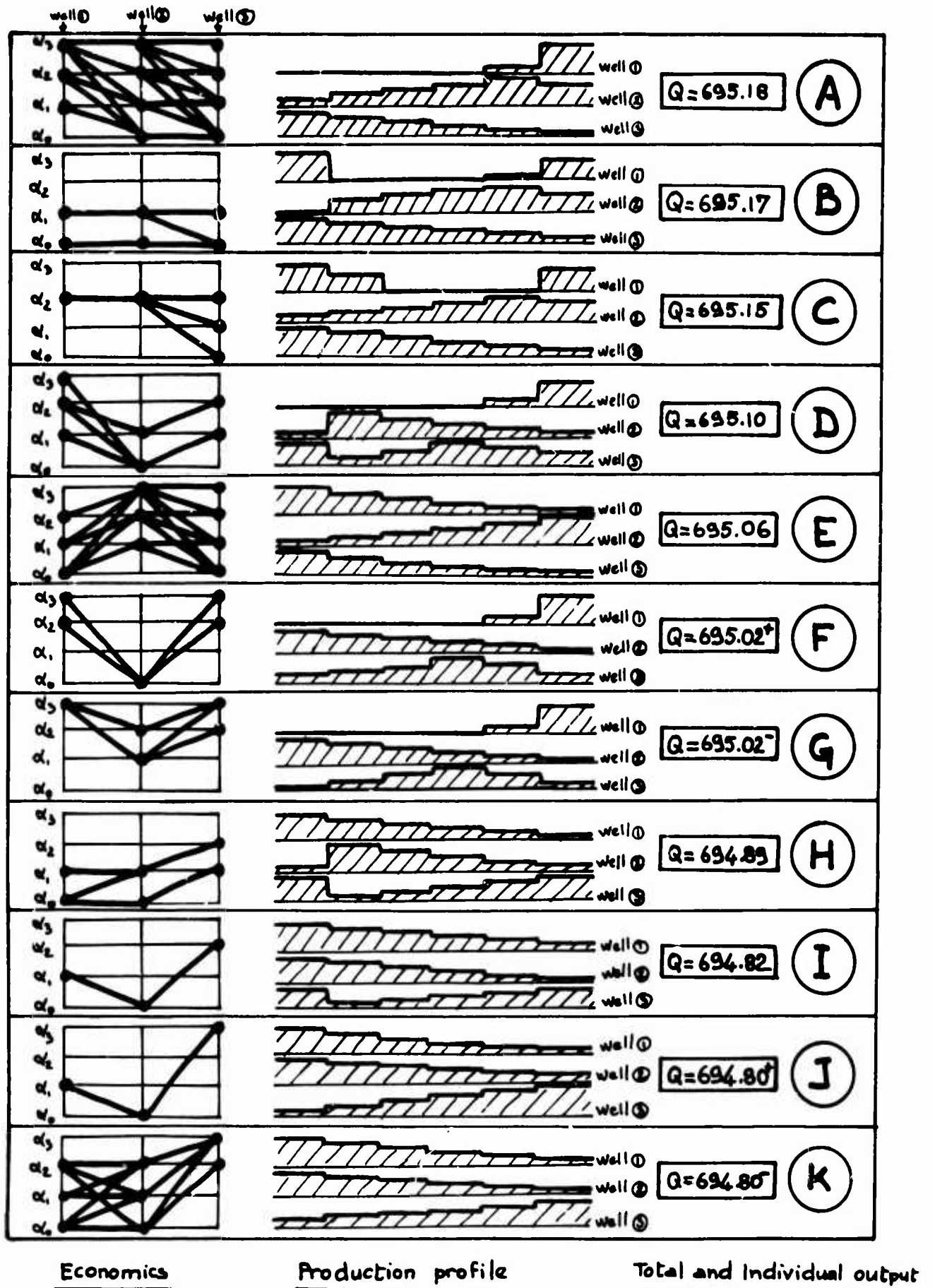


Figure: 7

Total bounded output - Inequality cases $n=6$

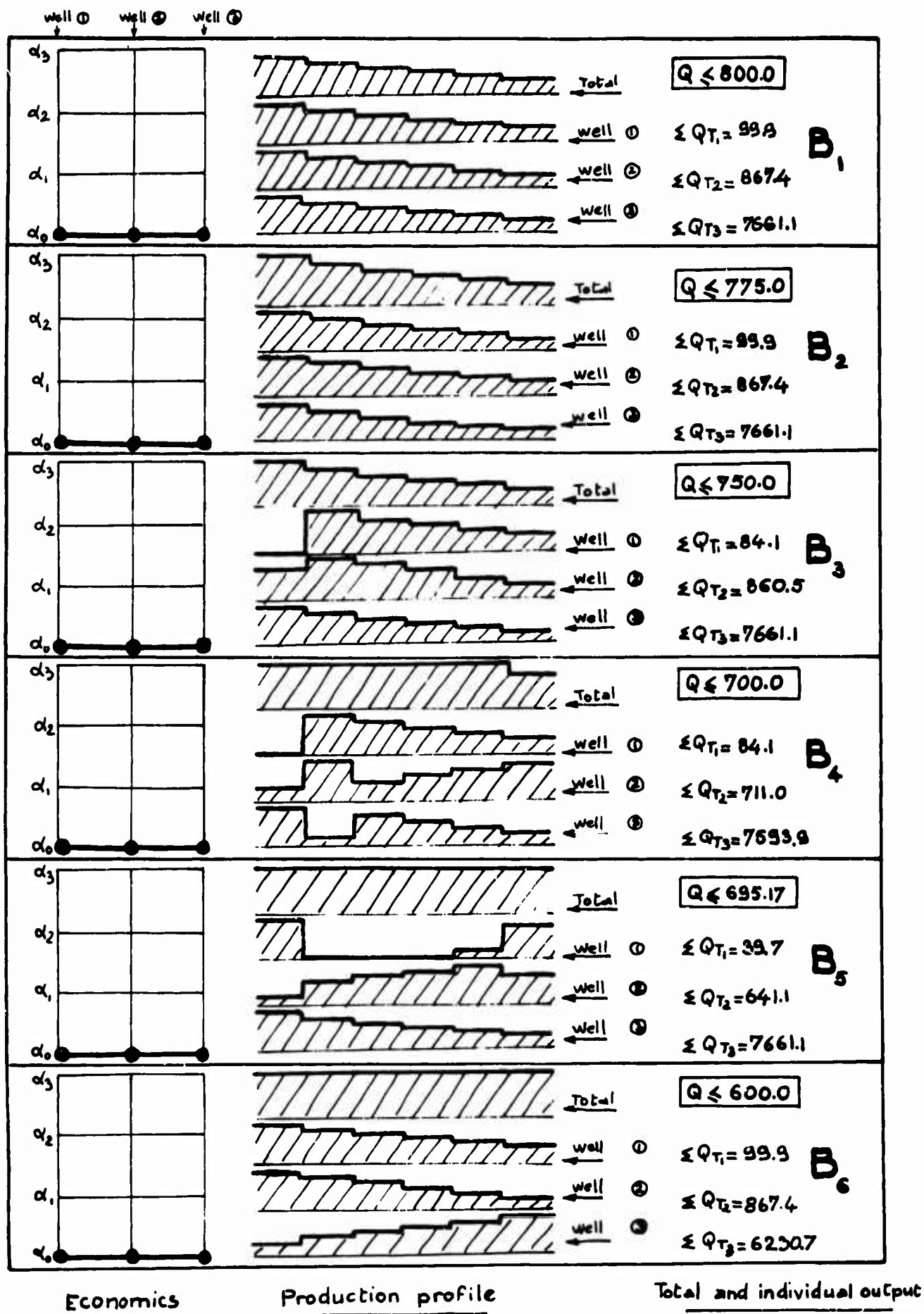


Figure: 8

Total bounded output - Inequality cases $n=6$

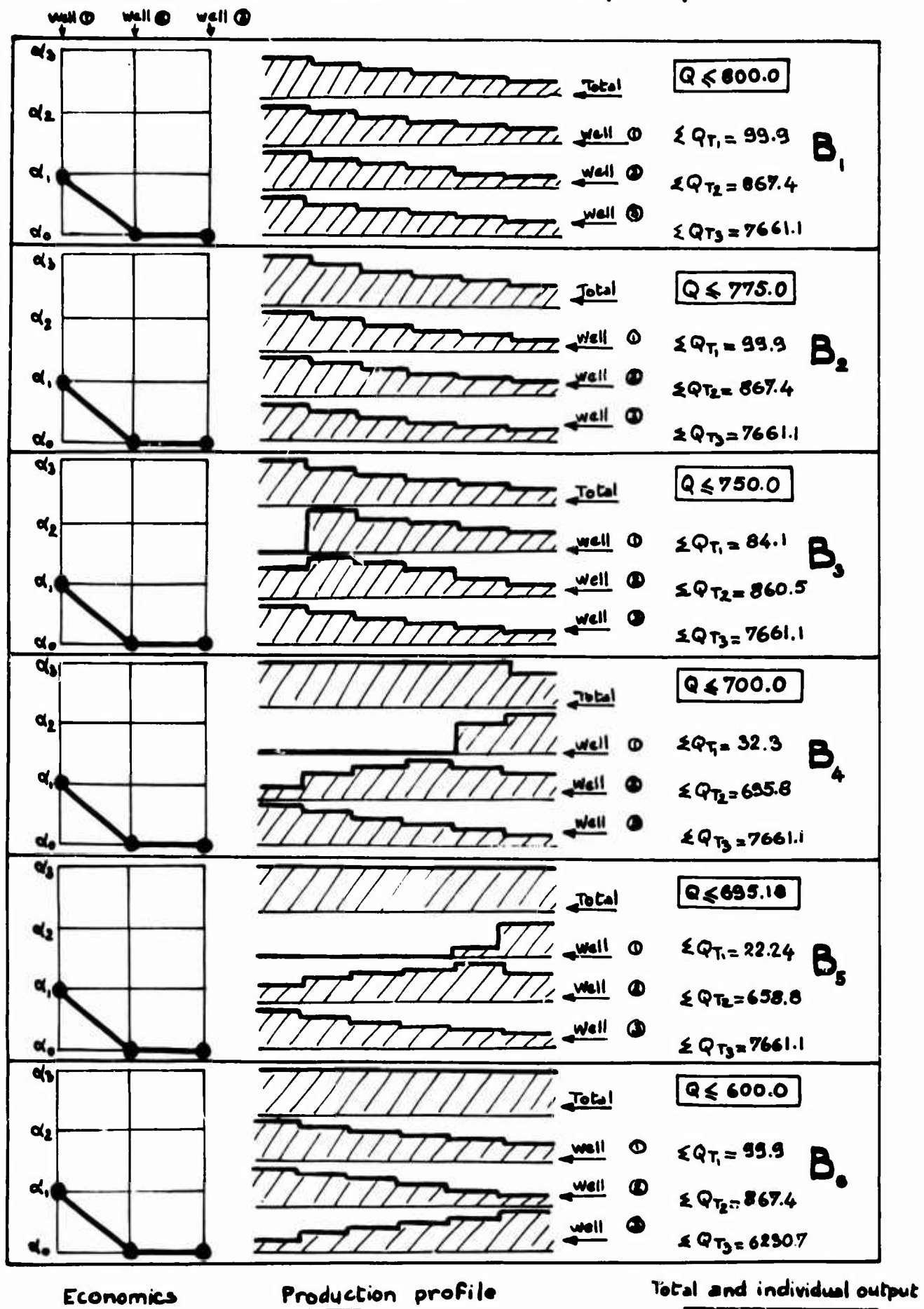
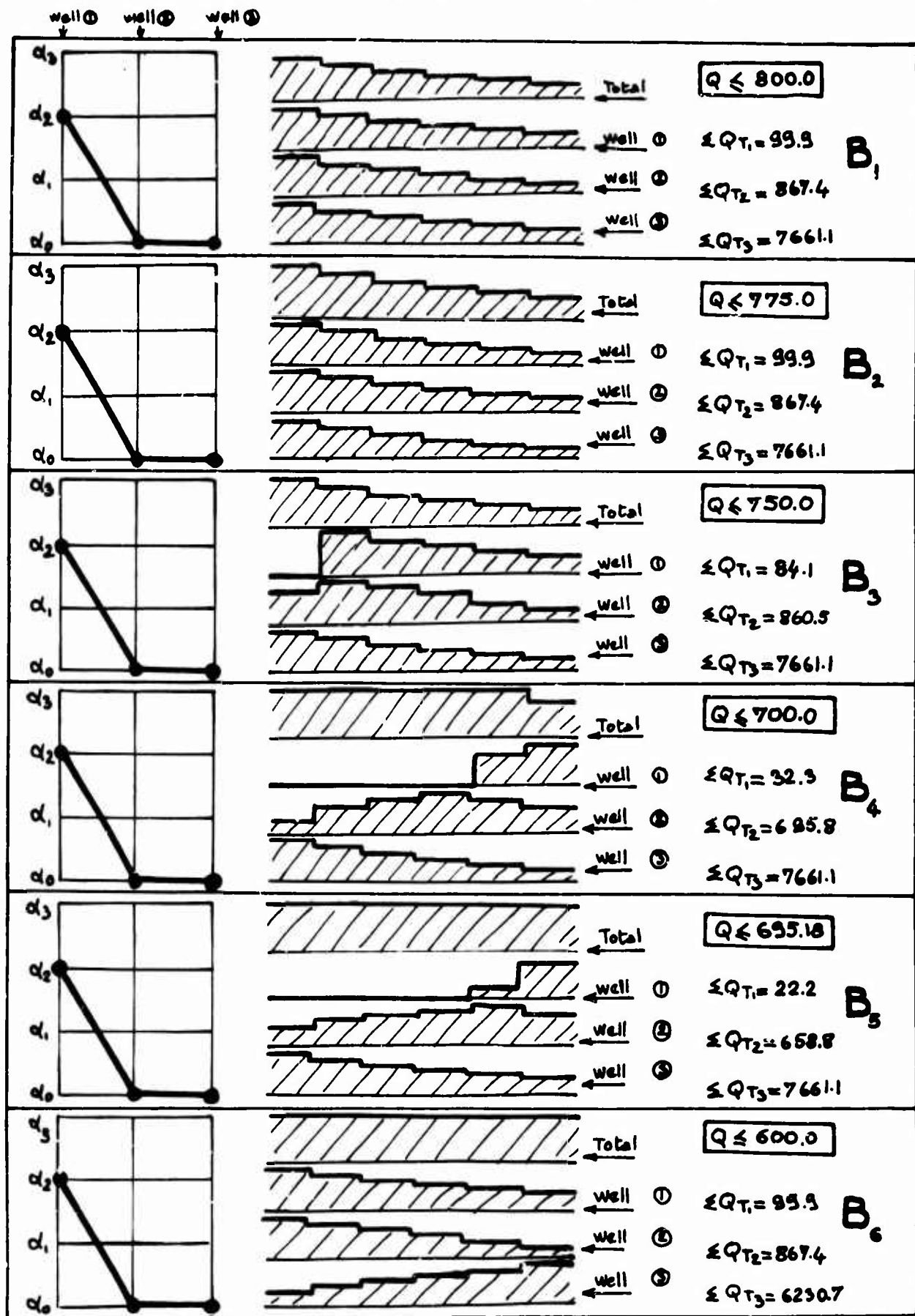


Figure: 9

Total bounded output - Inequality cases $n=6$



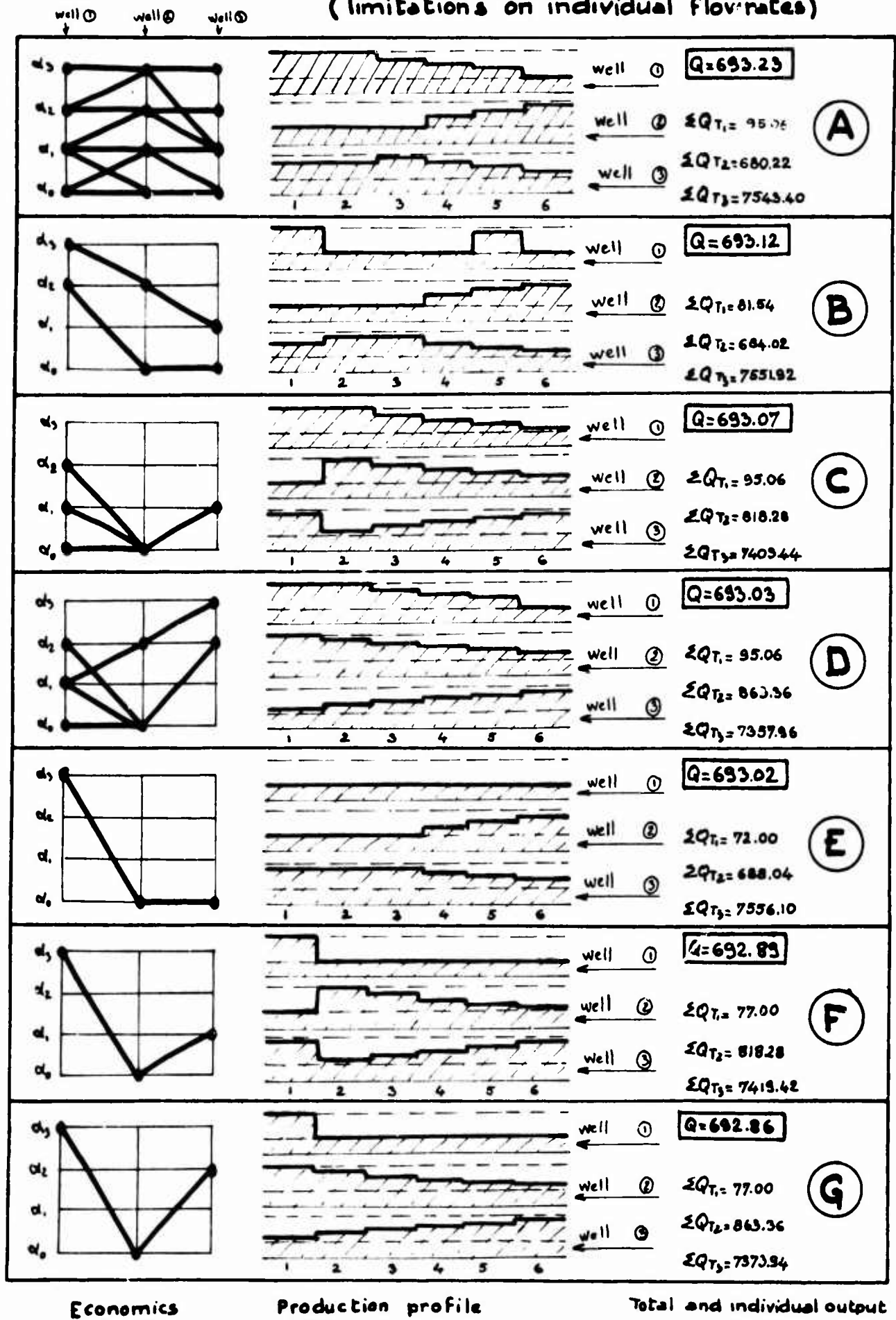
Economics

Production profile

Total and individual output

figure: 10

Total constant output - Equality cases $n=6$
(limitations on individual flow rates)



Economics

Production profile

Total and individual output

Figure: 11

Total constant output - Equality cases $n=12$

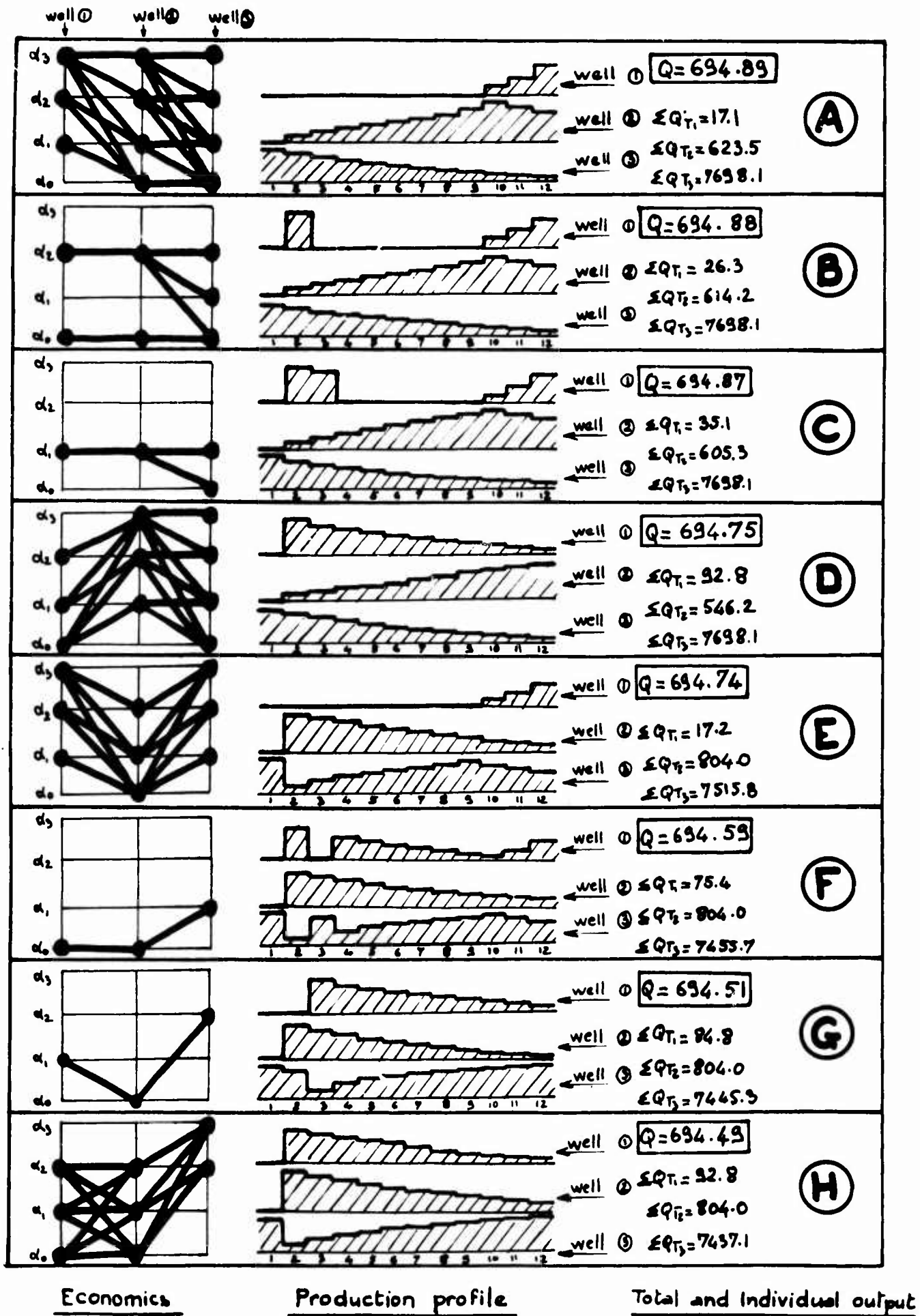


Figure: 12

Total bounded output - Inequality cases $n=12$

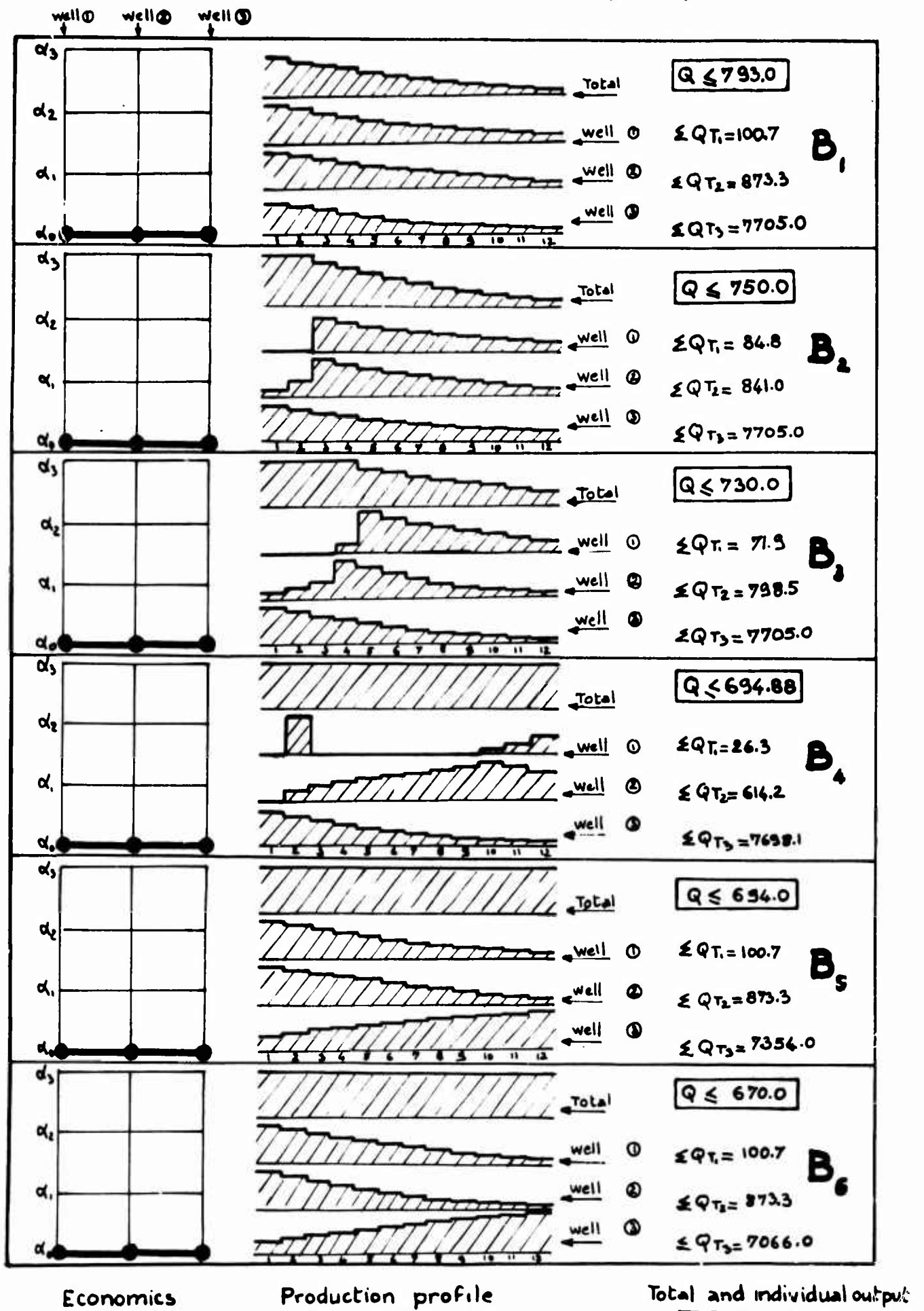


Figure: 13

Neyman-Pearson lemma

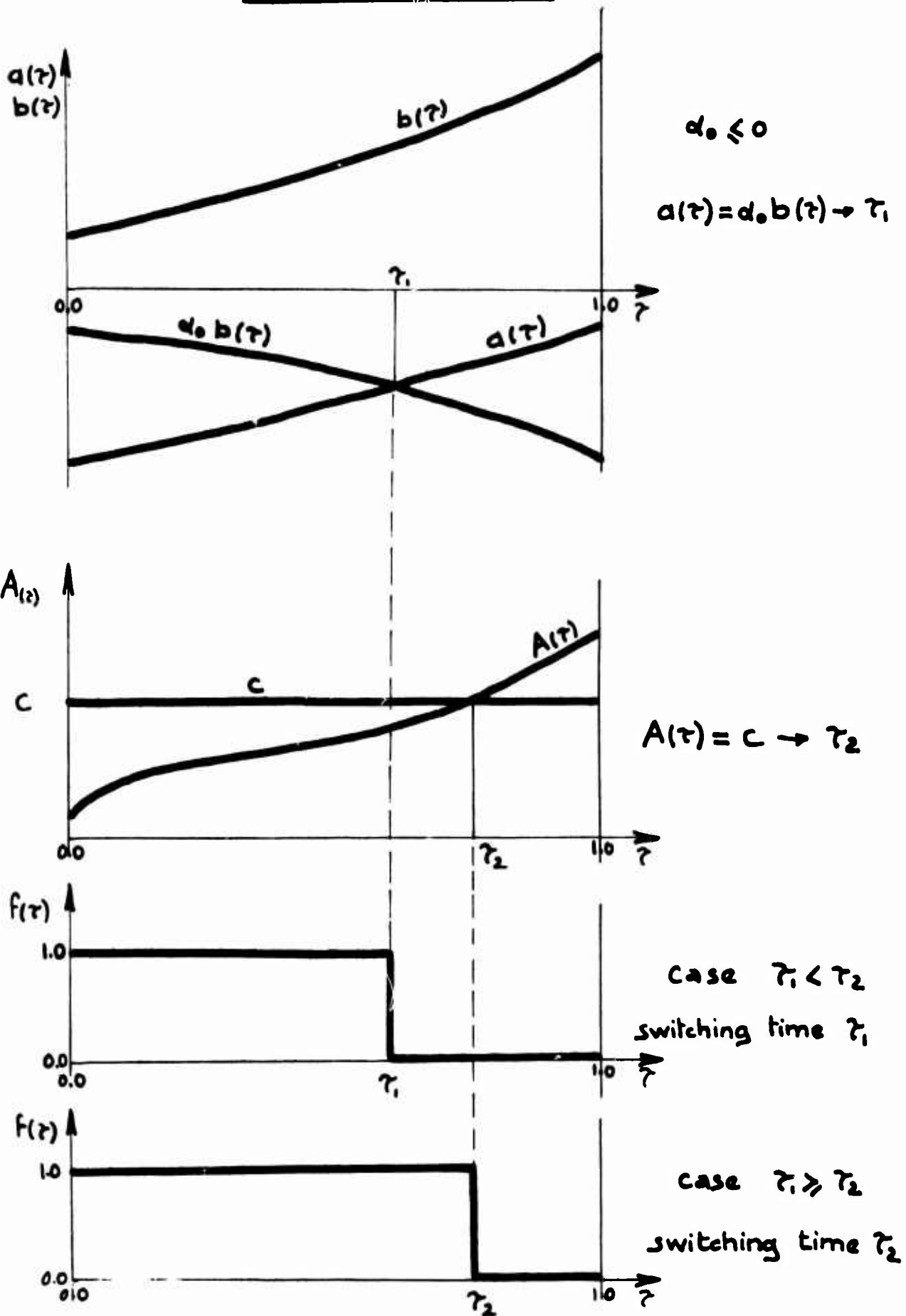


Figure: 14

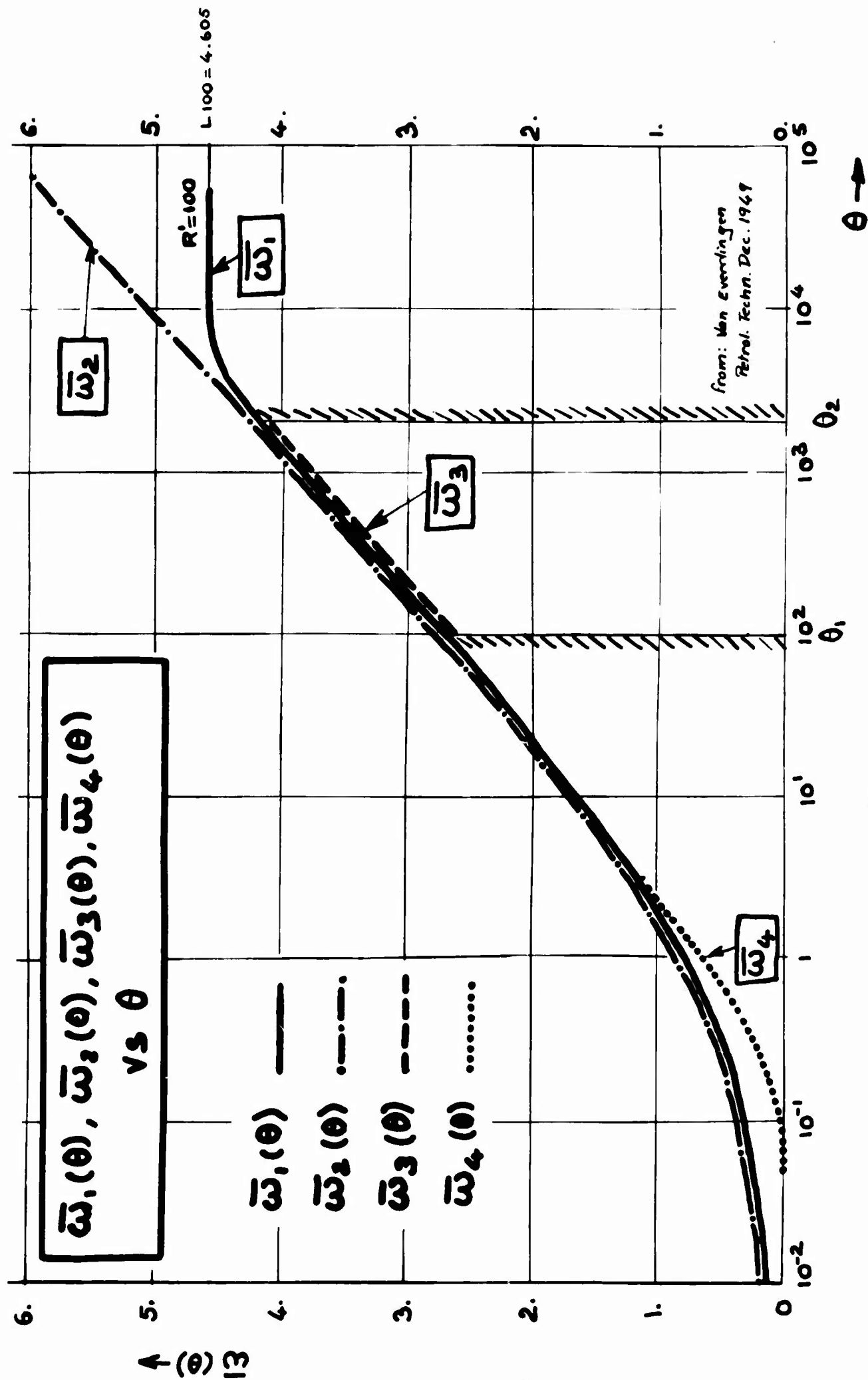


Figure : 15

$\bar{\omega}_1(\theta)$ vs θ $3.0 \leq R' \leq 1000$
 $\bar{\omega}_2(\theta)$ vs θ $R' = \infty$

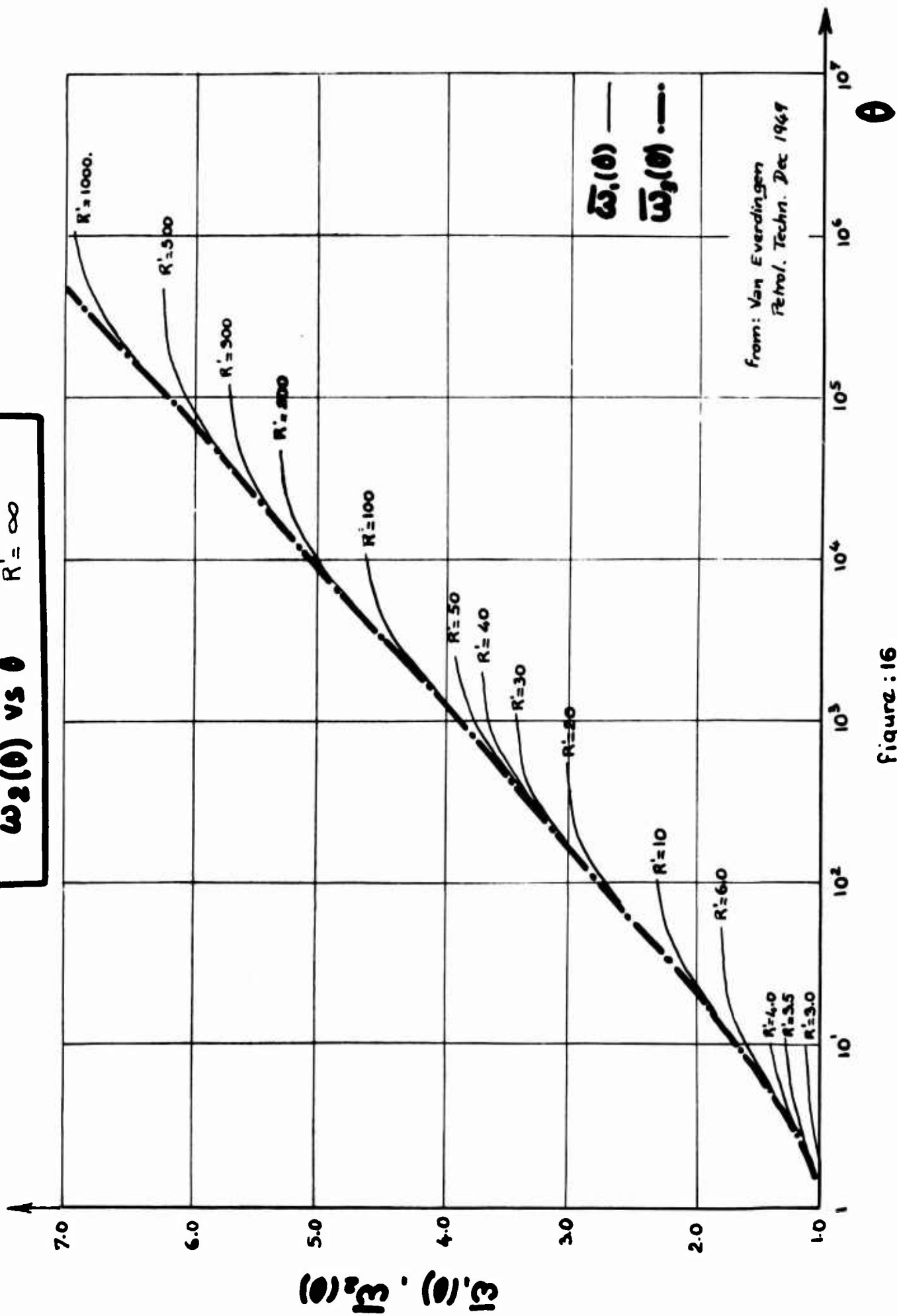


Figure : 16

$\frac{\bar{\omega}_1(\theta) - \bar{\omega}_3(\theta)}{\bar{\omega}_1(\theta)} \% \text{ vs } \theta$

$R' > 100$

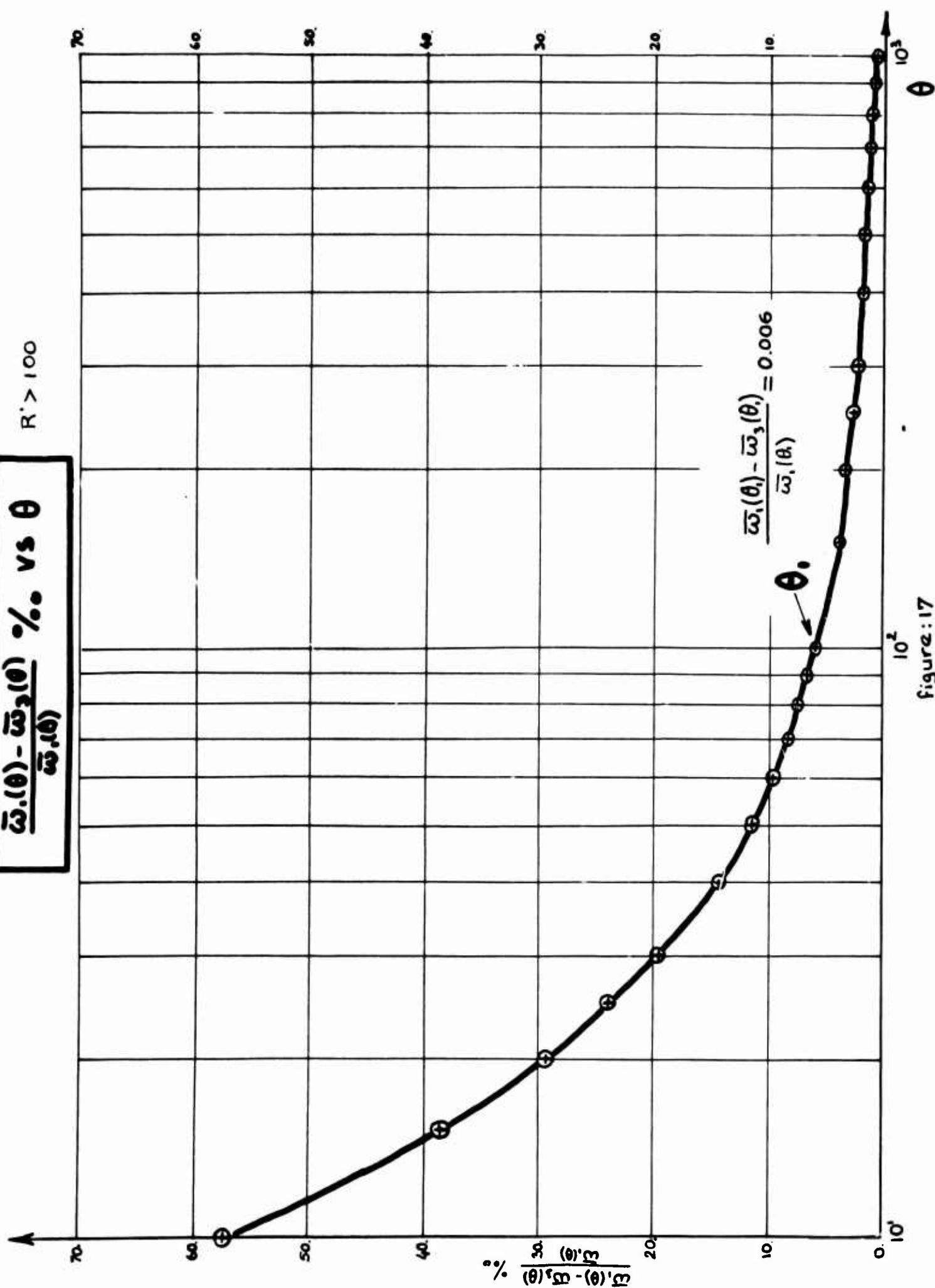


Figure: 17

$$\theta_2 \text{ vs } R'$$

$$\frac{\bar{\omega}_1(\theta) - \bar{\omega}_3(\theta)}{\bar{\omega}_1(\theta)} = \text{cst}$$

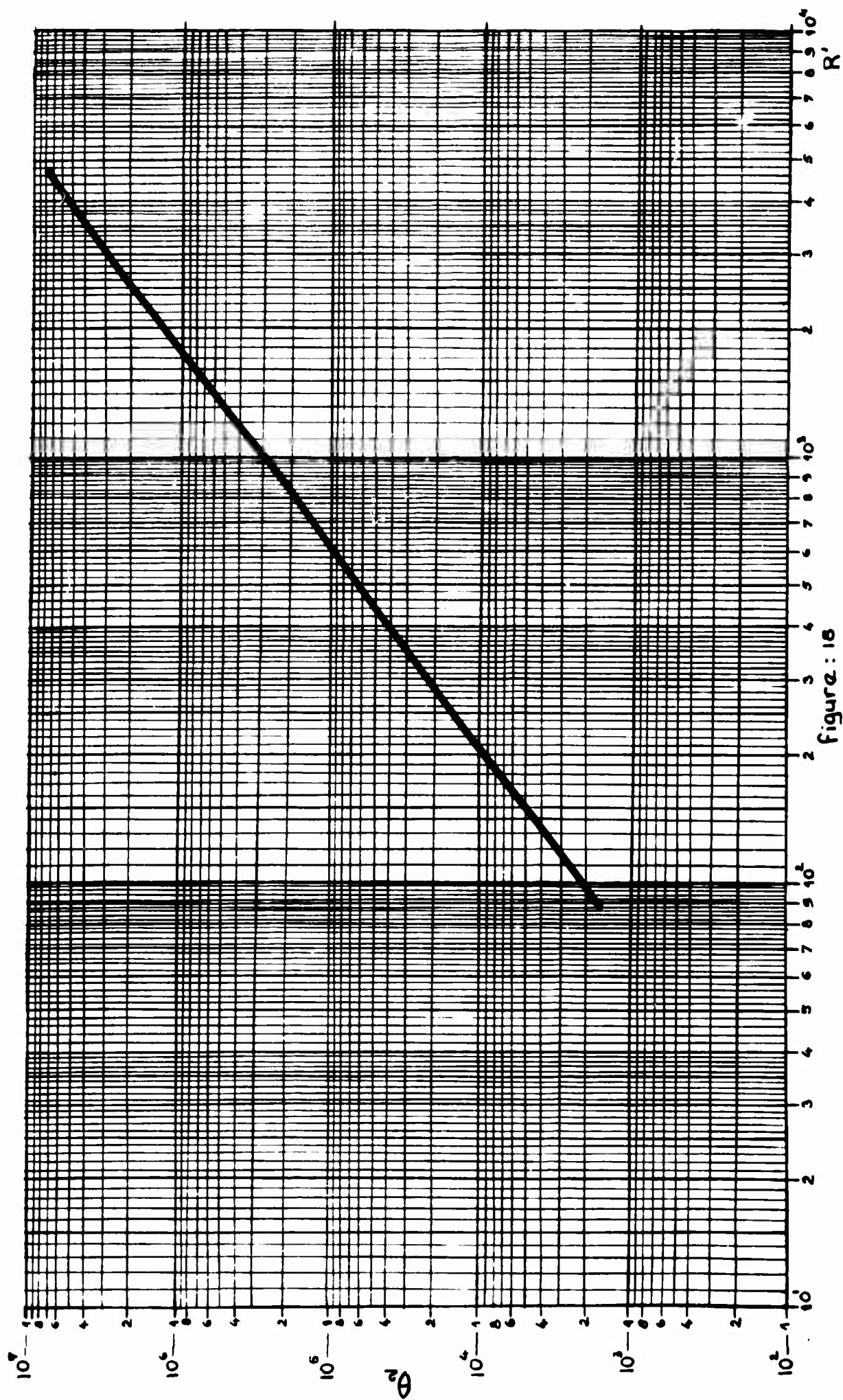


Figure: 18

α_n vs n

$R' \geq 100$

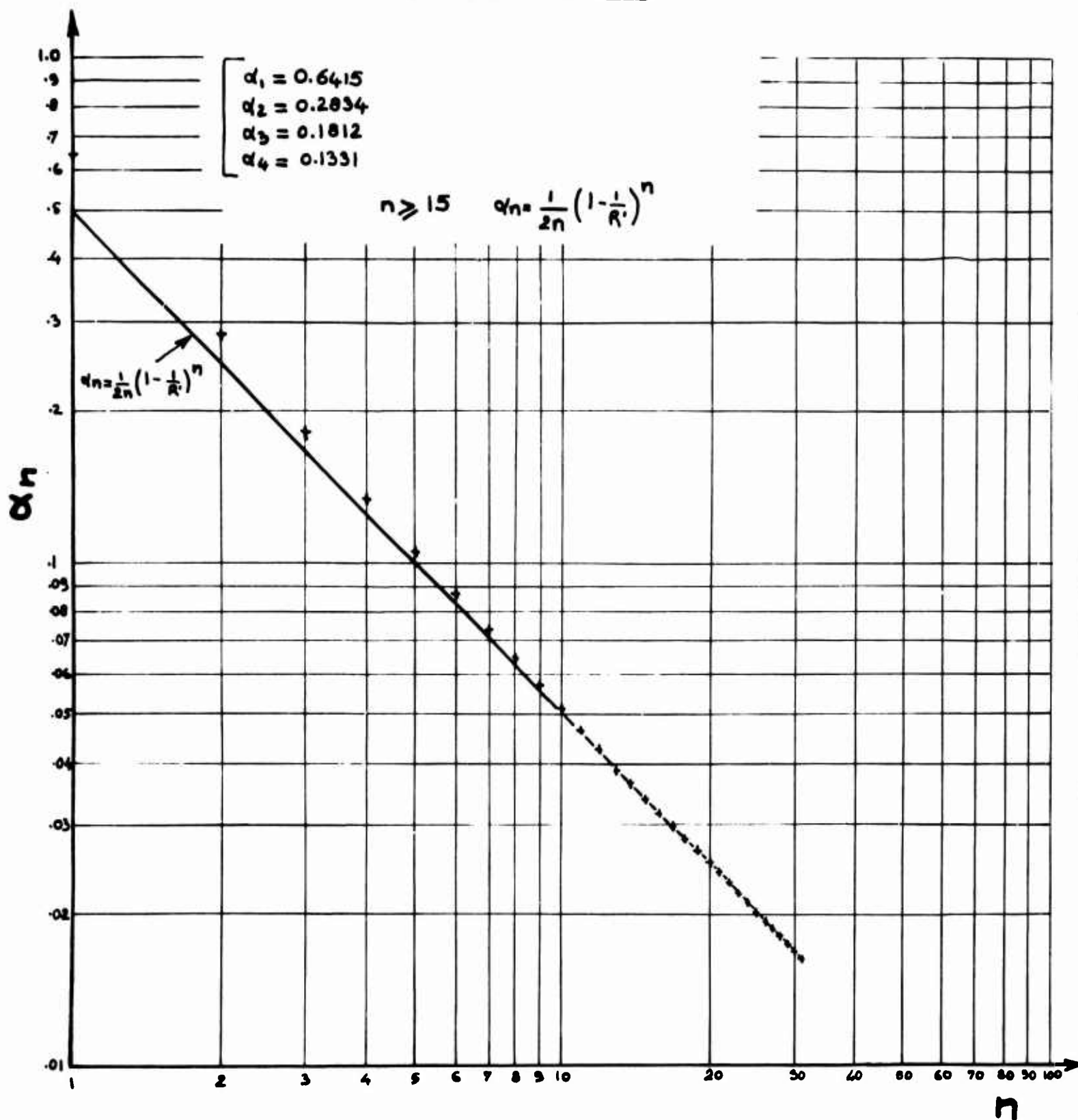


Figure : 19

$\beta_n R'$ vs n

$R' \geq 100$

$[\beta_n R' | J_0(\beta_n R') = 0]$

$n \geq 15 \quad \beta_n R' = \pi(n - \frac{1}{4})$

$\beta_1 R' = 2.405$
 $\beta_2 R' = 5.520$
 $\beta_3 R' = 8.654$
 $\beta_4 R' = 11.79$

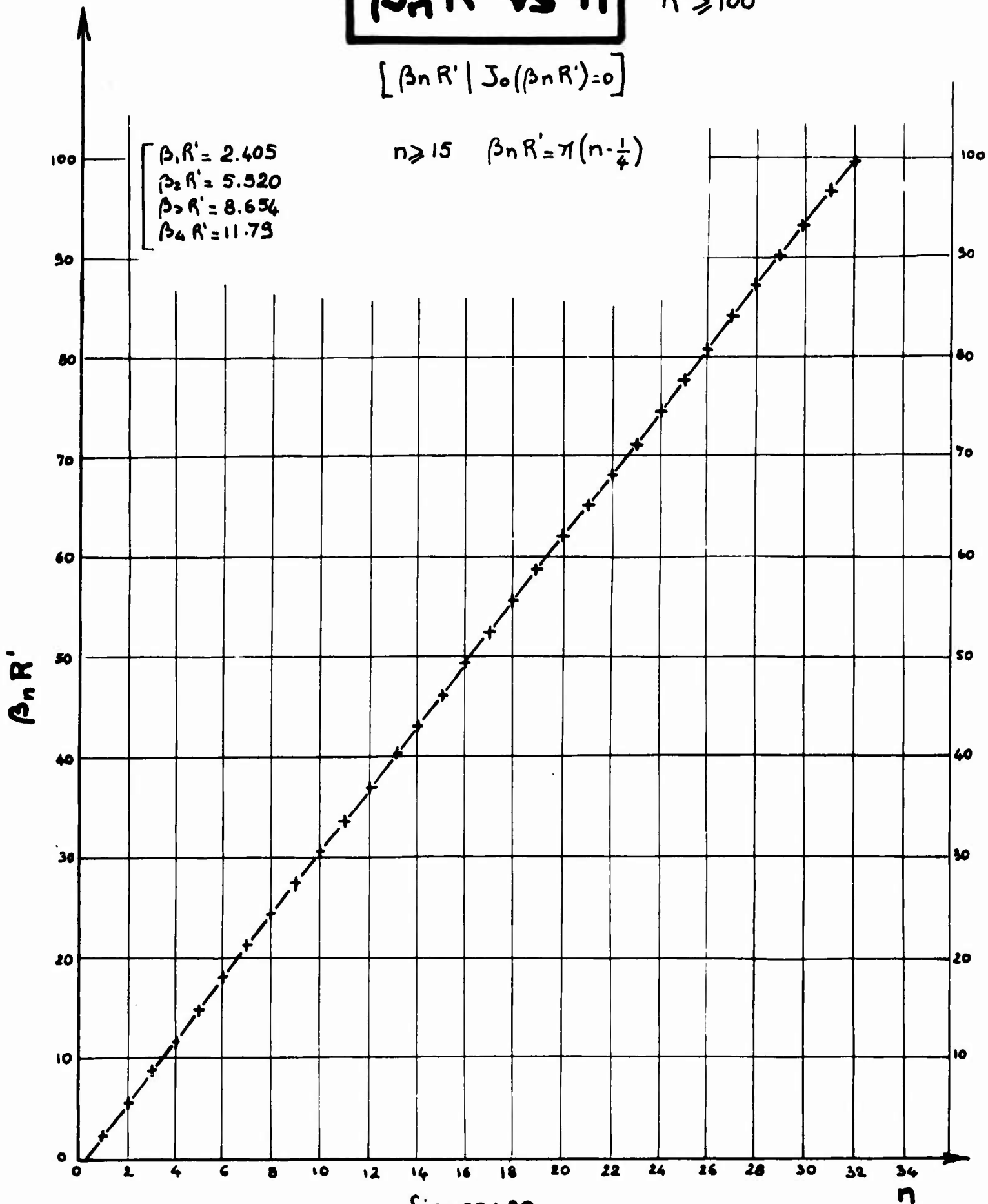


Figure: 20